arXiv:1504.02746v1 [math.PR] 10 Apr 2015

Concentration of the invariant measures for the periodic Zakharov, KdV, NLS and Gross–Piatevskii equations in $1D$ and $2D$

Gordon Blower
Department of Mathematics and Statistics
Lancaster University
Lancaster, LA1 4YF
England UK
email: g.blower@ lancaster.ac.uk
tel: +44 01524 593962
10th April 2015

Abstract This paper concerns Gibbs measures ν for some nonlinear PDE over the D-torus $\mathbf{T}^D$. The Hamiltonian $H = \int_{\mathbf{T}^D} \|\nabla u\|^2 - \int_{\mathbf{T}^D} |u|^p$ has canonical equations with solutions in $\Omega_N = \{u \in L^2(\mathbf{T}^D) : \int |u|^2 \leq N\}$. For $D = 1$ and $2 \leq p < 6$, Ω_N supports the Gibbs measure $\nu(du) = Z^{-1} e^{-H(u)} \prod_{x\in\mathbf{T}} du(x)$ which is normalized and formally invariant under the flow generated by the PDE. The paper proves that $(\Omega_N, \|\cdot\|_{L^2}, \nu)$ is a metric probability space of finite diameter that satisfies the logarithmic Sobolev inequalities for the periodic KdV, the focussing cubic nonlinear Schrödinger equation and the periodic Zakharov system. For suitable subset of Ω_N, a logarithmic Sobolev inequality also holds in the critical case $p = 6$. For $D = 2$, the Gross–Piatevskii equation has $H = \int_{\mathbf{T}^2} \|\nabla u\|^2 - \int_{\mathbf{T}^2} (V * |u|^2)|u|^2$, for a suitable bounded interaction potential V and the Gibbs measure ν lies on a metric probability space $(\Omega, \|\cdot\|_{H^{-s}}, \nu)$ which satisfies LSI. In the above cases, (Ω, d, ν) is the limit in L^2 transportation distance of finite-dimensional $(\Omega_n, \|\cdot\|, \nu_n)$ given by Fourier sums.

Keywords Gibbs measure, logarithmic Sobolev inequality transportation
Classification: 37L55; 35Q53

1. Introduction

The periodic Korteweg–de Vries and cubic nonlinear Schrödinger equations in space dimension D may be realised as Hamiltonian systems with an infinite-dimensional phase space $L^2(\mathbf{T}^D, \mathbf{R})^{\times 2}$. For instance, the Hamiltonian

$$H_p(u) = \frac{1}{2} \int_{\mathbf{T}^D} \|\nabla u(\theta)\|^2 \frac{d^D\theta}{(2\pi)^D} - \frac{\lambda}{p} \int_{\mathbf{T}^D} |u(\theta)|^p \frac{d^D\theta}{(2\pi)^D}, \tag{1.1}$$

is focussing for $\lambda > 0$ and defocussing for $\lambda < 0$, and the canonical equations generate the NLS. The critical exponent for existence of smooth solutions over all time is $p = 2 + (4/D)$ by [9, p. 6]. In particular H_4 generates the cubic NLS equation for the field u. For $N > 0$, traditionally called the number operator [15], let Ω_N be the

$$\Omega_N = \Big\{ u \in L^2(\mathbf{T}^D; \mathbf{C}) : \int_{\mathbf{T}^D} |u(\theta)|^2 \frac{d^D\theta}{(2\pi)^D} \leq N \Big\}. \tag{1.2}$$

Observe that Ω_N is formally invariant under the flow generated by (1.1).

For $D = 1$, Lebowitz, Rose and Speer [15] introduced an associated Gibbs ν measure and determined conditions under which ν can be normalized to define a probability measure on Ω_N; thus they introduced the modified canonical ensemble as the metric probability space $\mathbf{X} = (\Omega_N, \|\cdot\|_{L^2}, \nu)$. The purpose is to have a statistical mechanical model of typical solutions of KdV and NLS, not just the smooth solutions. In this paper, we describe concentration of Gibbs measures in terms of logarithmic Sobolev inequalities, and then use Sturm's theory of metric measure spaces [19] to obtain convergence of Gibbs measures on finite-dimensional phase spaces to the true Gibbs measure.

Definition ($LSI(\alpha)$) Let (X, d) be a complete and separable metric space, which is a length space with no isolated points, and μ a probability measure on X. For $f : X \to \mathbf{R}$, introduce the norm of the gradient $|\nabla f(x)| = \limsup_{y\to x} |f(y) - f(x)|/d(x, y)$. Then (X, d, μ) satisfies the logarithmic Sobolev inequality with constant $\alpha > 0$ (abbreviated $LSI(\alpha)$) if

$$\int_X f(x)^2 \log\Big(f(x)^2/\int_X f^2 d\mu\Big)\mu(dx) \leq \frac{2}{\alpha}\int_X \big|\nabla f\big|^2 \mu(dx) \tag{1.3}$$

for all $f \in L^2(\mu; X; \mathbf{R})$ such that $|\nabla f(x)| \in L^2(\mu; X; \mathbf{R})$. See [21, chapter 21].

When $(X, d) = (\mathbf{R}^m, \|\cdot\|_E)$ for some Banach space norm E and $f : \mathbf{R}^m \to \mathbf{R}$ is continuously differentiable, then we have $|\nabla f(x)| = \|\nabla f(x)\|_{E^*}$, where ∇f is the usual gradient and E^* the dual normed space. In the analysis below, we generally apply $LSI(\alpha)$ to functions which may be expressed in terms of the Fourier coordinates, and we require inequalities with constants that do not depend directly upon the dimension of the phase space. Our results are closely related to those of [14], since LSI implies a spectral gap inequality by [21, Theorem 22.28].

Bourgain [6] showed that the Gibbs measure on suitably normalized subspaces could be constructed from random Fourier series, so that the Fourier coefficients give an explicit system of canonical coordinates for the phase space. Let $\mathrm{H}^s(\mathbf{T}^D) = \{\sum_{k\in\mathbf{Z}^D} a_k e^{ik\cdot\theta} : |a_0|^2 + \sum_{k\in\mathbf{Z}^d\setminus\{0\}} |k|^{2s}|a_k|^2 < \infty\}$. Let $(\gamma_k, \gamma_k')_{k\in\mathbf{Z}^D}$ be mutually independent standard Gaussian random variables. Then for $\rho > 0$, the periodic Brownian motion

$$b(\theta) = \sum_{k\in\mathbf{Z}^D} \frac{(\gamma_k + i\gamma_k')e^{ik\cdot\theta}}{\sqrt{\rho + |k|^2}} \qquad (\theta = (\theta_1, \ldots, \theta_D)) \tag{1.4}$$

lies in $\mathrm{H}^s(\mathbf{T}^D)$ almost surely for $s < 1 - (D/2)$.

For $D = 1$, Lebowitz, Rose and Speer [15] showed that for all $N < \infty$ and $2 \leq p < 6$ one can introduce $Z = Z(N, p, \lambda) > 0$ to normalize the Gibbs measure

$$\nu_N(du) = Z^{-1}\mathbf{I}_{\Omega_N}(u)e^{-H_p(u)} \prod_{\theta\in\mathbf{T}} du(\theta) \tag{1.5}$$

as a probability on Ω_N. However, for $p > 6$, so such Z exists. See also [13, 16] for alternative constructions of the Gibbs measure.

In section 3 of this paper, we prove a logarithmic Sobolev inequality for ν_N when $D = 1$ and $p = 4$. The proof depends upon convexity of the Hamiltonian on Ω_N, and uses a criterion that originates with Bakry and Emery [2, 21]. In section 4, we deduce similar results for the periodic Zakharov system. In section 5, we use a similar method to prove a LSI for $u \in L^2(\mathbf{T}; \mathbf{R})$ and $p = 3$, where the Hamiltonian generates the KdV equation. For $D = 1$ and $p = 6$, there exists $N_0 > 0$ such that the Gibbs measure can be normalized on Ω_N for $N < N_0$, but not for $N > N_0$. In section 6, we obtain a logarithmic Sobolev inequality for subsets $\Omega_{N,\kappa} = \{u \in \Omega_N : \|u\|^2_{\mathrm{H}^s} \leq \kappa\}$ and $1/4 < s < 1/2$ which support most of the Gibbs measure. While these Gibbs measures are absolutely continuous with respect to Brownian loop, the Radon–Nikodym derivatives are not logarithmically concave, so our results do not follow directly from the curvature computations in [19]. Instead we use uniform convexity of the Hamiltonians on suitable Ω_N, and exploit the property that LSI are stable under suitable perturbations; see [21, Remark 21.5].

The partial sums of the spatial Fourier series suggest classical Hamiltonians on finite-dimensional phase spaces X^n given by the low wave numbers, which generate autonomous systems of ordinary differential equations in the canonical coordinates. Such X^n support Liouville measures ν_n, which are invariant under the flow generated by the canonical equations, and which give metric probability spaces $\mathbf{X}^n = (X^n, \|\cdot\|_{\mathbf{R}^{2n}}, \nu_n)$. We show that for $D = 1$ and $p \leq 6$, the $\mathbf{X}^n$ converge as metric probability spaces to $\mathbf{X}$ in the L^2 transportation distance; this extends the notion of approximating the solution of a PDE by Fourier partial sums.

The lack of smoothness of $b(\theta)$ complicates the analysis of the NLS equation in two dimensions, and more drastically in higher space dimensions. The integral (1.1) with $p = 4$ is critical for existence of invariant measures in the 2D focussing case. So one introduces a real interaction potential V and works with the Gross–Piatevskii equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial \theta_1^2} + \frac{\partial^2 u}{\partial \theta_2^2} + \lambda\big(V * |u|^2\big)u = 0, \tag{1.6}$$

which is also credited to Hartree. In section 7, we impose additional hypotheses including $V \in L^\infty(\mathbf{T}^2; \mathbf{R})$ to obtain a finite-dimensional logarithmic Sobolev inequality and then $V \in \mathrm{H}^{1+2s}(\mathbf{T}^2; \mathbf{R})$ to obtain a infinite-dimensional LSI. We regard this as realistic, since in their model of a supersolid, Pomeau and Rica [17] consider a soft sphere interaction with V bounded. The Gibbs measure is supported on distributions in H^{-s}, so the solutions of (1.6) are typically not in $L^2(\mathbf{T}^2; \mathbf{C})$. Nevertheless, in section 8 we achieve convergence in L^2 transportation distance for finite-dimensional metric probability spaces towards Gibbs measure on the phase space for the PDE.

2 Metric Measure Spaces for Trigonometric Systems

Sturm [19] has developed a theory of metric measure spaces which refines the metric geometry of Gromov and Hausdorff. We recall some definitions, which simplify slightly in our setting of probability spaces, which Sturm calls normalized measure spaces.

Let (X, d) be a complete and separable metric space. Now let $\mathrm{Prob}_0(X)$ be the space of Radon probability measures on (X, d) with the weak topology; a metric probability space $\hat{X}$ consists of (X, d, μ) with $\mu \in \mathrm{Prob}_0(X)$. Suppose that $\mu, \nu \in \mathrm{Prob}_0(X)$ and that ν is absolutely continuous with respect to μ and that $f = \frac{d\nu}{d\mu}$ is the Radon–Nikodym derivative. Then the relative entropy of ν with respect to μ is

$$\mathrm{Ent}(\nu \mid \mu) = \int_X f(x) \log f(x) \, \mu(dx), \tag{2.1}$$

so that $0 \leq \mathrm{Ent}(\nu \mid \mu) \leq \infty$. For $1 \leq s < \infty$, $\mathrm{Prob}_s(X)$ consists of the subspace of $\mu \in \mathrm{Prob}_0(X)$ such that $\int_X \delta(x_0, x)^s \mu(dx) < \infty$ for some or equivalently all $x_0 \in X$. The Wasserstein distance of order s between $\mu, \nu \in \mathrm{Prob}_s(X)$ is

$$\mathrm{W}_s(\mu, \nu) = \inf_\pi \Big\{ \Big(\iint_{X \times X} \delta(x, y)^s \pi(dxdy) \Big)^{1/s} : \pi_1 = \mu, \pi_2 = \nu \Big\} \tag{2.2}$$

where $\pi \in \mathrm{Prob}_s(X \times X)$ with marginals $\pi_1 = \mu$ and $\pi_2 = \nu$ is called a transportation plan, and δ^s is the cost function. Then $(\mathrm{Prob}_s(X), \mathrm{W}_s)$ is a metric space.

Suppose further that there exists $\alpha > 0$ such that

$$\mathrm{W}_s(\nu, \mu) \leq \sqrt{\frac{2}{\alpha} \mathrm{Ent}(\nu \mid \mu)} \tag{2.3}$$

for all $\nu \in \mathrm{Prob}_s(X)$ that are of finite relative entropy with respect to μ. Then μ is said to satisfy the transportation inequality $T_s(\alpha)$. We repeatedly use the result of Otto and Villani that $LSI(\alpha)$ implies $T_2(\alpha)$ on Euclidean space; see [21, 22.17].

Definition (L^2 transportation distance) A pseudo metric on a nonempty set Z is a function $\delta : Z \times Z \to [0, \infty]$ that is symmetric, vanishes on the diagonal, and satisfies the triangle inequality. A coupling of pseudo metric spaces (X, δ_1) and (Y, δ_2) is a pseudo metric space (Z, δ) such that $Z = X \sqcup Y$ and $\delta|_{X \times X} = \delta_1$ and $\delta|_{Y \times Y} = \delta_2$. Given metric probability spaces $\hat{X} = (X, \delta_1, \mu_1)$ and $\hat{Y} = (Y, \delta_2, \mu_2)$, consider a coupling δ of these metric spaces and $\pi \in \mathrm{Prob}_0(X \times Y)$ with marginals μ_1 and μ_2. Then the L^2 transportation distance is

$$\mathrm{D}_{L^2}(\hat{X}, \hat{Y}) = \inf_{\delta, \pi} \Big\{ \Big(\iint_{X \times Y} \delta(x, y)^2 \pi(dxdy) \Big)^{1/2} \Big\}, \tag{2.4}$$

where the infimum is taken over all such couplings δ and all transportation plans π. One can easily show that if $\mu_1 \in \mathrm{Prob}_2(X)$ and $\mu_2 \in \mathrm{Prob}_2(Y)$, then $\mathrm{D}_{L^2}(\hat{X}, \hat{Y}) < \infty$. The diameter of $\hat{X}$ is $\sup\{d(x, y) : x, y \in \mathrm{support}(\mu)\}$. The family of isomorphism classes of metric probability spaces that have finite diameter gives a metric space $(\mathbf{X}, \mathrm{D}_{L^2})$ by results of [19].

To obtain $LSI(\alpha)$ for measures on Hilbert space from their finite-dimensional marginals, we use the following Lemma, which is related to Theorem 1.3 from [4].

Lemma 2.1 *Let $d\nu = e^{-V(x)} \prod_{j=1}^{\infty} dx_j$ be a Radon probability measure on $\ell^2(\mathbf{N};\mathbf{R})$, and let $\mathcal{F}_n$ be σ-algebra that is generated by the first n coordinate functions, and let ν_n be the marginal of ν for the first n coordinates. Suppose that*

(i) V is continuously differentiable, and $\int \|\nabla V(x)\|_{\ell^2}^2 \nu(dx) < \infty$;

(ii) there exists $\alpha > 0$ such that $LSI(\alpha)$ holds for $X^n = (\mathbf{R}^n, \|\cdot\|_{\ell^2}, \nu_n)$ for all n.

Then $LSI(\alpha)$ holds for $X^\infty = (\ell^2, \|\cdot\|_{\ell^2}, \nu)$, and $X^n \to X^\infty$ in D_{L^2} as $n \to \infty$.

Proof. For $0 \leq f \in L^2(\ell^2; \nu; \mathbf{R})$, let $f_n = \mathbf{E}(f \mid \mathcal{F}_n)$, so that $0 \leq f_n$ and $f_n \to f$ almost surely and in L^2 as $n \to \infty$ by the martingale convergence theorem. By Jensen's inequality applied to the convex function $\varphi(x) = x^2 \log x^2$ for $x > 0$, we have

$$\int f_n^2 \log_+ f_n^2 \, d\nu - \int f_n^2 \log_- f_n^2 \, d\nu \leq \int f^2 \log_+ f^2 \, d\nu - \int f^2 \log_- f^2 \, d\nu. \tag{2.5}$$

Now $\varphi(x) \geq -1/e$, so we can apply the dominated convergence theorem to the terms with $\log_-$ and Fatou's lemma to the positive terms with $\log_+$ to deduce that the entropy term on the left-hand side of LSI satisfy

$$\begin{aligned}\int f^2 \log\Big(f^2 / \int f^2 d\nu\Big) d\nu &= \lim_{n\to\infty} \int f_n^2 \log\Big(f_n^2 / \int f_n^2 d\nu\Big) d\nu \\ &\leq \limsup_{n\to\infty} \frac{2}{\alpha} \int_{\mathbf{R}^n} \|\nabla f_n(x)\|_{\ell^2}^2 \nu_n(dx). \end{aligned} \tag{2.6}$$

Integrating by parts in the first n coordinates, we see that $\nabla f_n = \mathbf{E}(\nabla f \mid \mathcal{F}_n) + \mathbf{E}((f_n - f)\nabla V \mid \mathcal{F}_n)$, so by the Cauchy–Schwarz inequality

$$\frac{2}{\alpha} \int \|\nabla f_n\|^2 d\nu_n \leq \frac{2(1+\varepsilon_n)}{\alpha} \int \|\nabla f\|^2 d\nu + \frac{2(1+\varepsilon_n)}{\alpha \varepsilon_n} \Big(\int |f_n - f|^2 d\nu\Big)^{1/2} \Big(\int \|\nabla V\|^2 d\nu\Big)^{1/2} \tag{2.7}$$

where we can choose $\varepsilon_n > 0$ decreasing to 0 so that (2.6) and (2.7) give

$$\int f^2 \log\Big(f^2 / \int f^2 d\nu\Big) d\nu \leq \frac{2}{\alpha} \int \|\nabla f\|^2 d\nu. \tag{2.8}$$

Hence $\hat{X}^\infty$ satisfies $LSI(\alpha)$. Now $LSI(\alpha)$ implies $T_1(\alpha)$ by [21, 22.17], so $\int \exp(\alpha \|x\|^2/2)\nu(dx) < \infty$. Any continuous and bounded function $f_n : \mathbf{R}^n \to \mathbf{R}$ may be identified with a function on the first n coordinates of ℓ^2, so the equation $\int f_n d\nu_n = \int f_n d\nu$ determines $\nu_n \in \mathrm{Prob}_2(\mathbf{R}^n)$. We write $x = (\xi_j)_{j=1}^\infty \in \ell^2$ as $x_n = (\xi_1, \ldots, \xi_n)$ and $x^n = (\xi_{n+1}, \xi_{n+2}, \ldots)$ and introduce $p_n(dx^n \mid \xi_n) \in \mathrm{Prob}_2(\ell^2)$ by disintegrating $\nu(dx) = p_n(dx^n \mid x_n)\nu_n(dx_n)$ with respect to ν_n; then we couple X^n with X^∞ by mapping $X^n \to X^\infty$ via $x_n \mapsto (x_n, 0)$. To transport ν_n to ν, we select x_n according to the law ν_n, then select x^n according to the law $p_n(dx^n \mid x_n)$; hence

$$\mathrm{D}_{L^2}(X^n, X^\infty)^2 \leq \iint_{\mathbf{R}^n \times \ell^2} \|x^n\|_{\ell^2}^2 p_n(dx^n \mid x_n)\nu_n(dx_n) = \int_{\ell^2} \|x - \mathbf{E}(x \mid \mathcal{F}_n)\|_{\ell^2}^2 \nu(dx), \tag{2.9}$$

which converges to zero as $n \to \infty$ by the dominated convergence theorem; so $X^n \to X^\infty$ in D_{L^2} as $n \to \infty$. $\square$

In subsequent sections, we introduce metric probability spaces relating to the trigonometric system over $\mathbf{T}^D$; their properties link curvature, dimension and the exponent in H. In space dimension D, let

$$X^n = \mathrm{span}\{e^{ik\cdot\theta} : k \in \mathbf{Z}^D; k = (k_1, \dots, k_D); |k_j| \leq n; j = 1, \dots, D\}, \tag{2.10}$$

so that $\iota_n : X^n \to X^{n+1}$ is the formal inclusion. When n is a dyadic power, the metric structure is well described by Littlewood–Paley theory. For $j \in \mathbf{N}$, we introduce the dyadic block $\Delta_j = \{2^{j-1}, 2^{j-1}+1, \dots, 2^j - 1\}$, and for $J = (j_1, \dots, j_D) \in \mathbf{N}^D$, let $\Delta(J) = \Delta_{j_1} \times \dots \times \Delta_{j_D}$. Let P_J be Dirichlet's projection onto the $\mathrm{span}\{e^{ik\cdot\theta} : k \in \Delta(J)\}$, and introduce the Hamiltonian

$$H_{\Delta(J)}(u) = \frac{1}{2}\int_{\mathbf{T}^D} \|\nabla P_J u(\theta)\|^2_{\ell^2} \frac{d^D\theta}{(2\pi)^D} - \frac{\lambda}{p}\int_{\mathbf{T}^D} \big|P_J u(\theta)\big|^p \frac{d^D\theta}{(2\pi)^D}. \tag{2.11}$$

Proposition 2.2 *For $2 \leq p \leq 2 + (4/D)$ and $N > 0$, there exists $\lambda > 0$ such that $H_{\Delta(J)}(u)$ is uniformly convex on Ω_N.*

Proof. We observe that $H_{\Delta(J)}$ is twice continuously differentiable on L^2, and

$$\begin{aligned}\Big(\frac{d^2}{dt^2}\Big)_{t=0} & H_{\Delta(J)}(u+tv) \\ &\geq \int_{\mathbf{T}^D} \|\nabla P_J v(\theta)\|^2_{\ell^2} \frac{d^D\theta}{(2\pi)^D} - \lambda(p-1)\int_{\mathbf{T}^D} \big|P_J u(\theta)\big|^{p-2}\big|P_J v(\theta)\big|^2 \frac{d^D\theta}{(2\pi)^D}.\end{aligned} \tag{2.12}$$

We write $|\Delta|$ for the cardinality of a finite set Δ, and observe that by the inequality of the means,

$$\sum_{\ell=1}^{D} |\Delta_{j_\ell}|^2 \geq D|\Delta(J)|^{2/D}. \tag{2.13}$$

Hence the first term on the right-hand side of (2.12) satisfies

$$\int_{\mathbf{T}^D} \|\nabla P_J v(\theta)\|^2_{\ell^2} \frac{d^D\theta}{(2\pi)^D} \geq \frac{D}{4}|\Delta(J)|^{2/D} \int_{\mathbf{T}^D} |P_J v(\theta)|^2 \frac{d^D\theta}{(2\pi)^D}. \tag{2.14}$$

Now we introduce de la Vallée Poussin's kernel K_J for $\Delta(J)$, so that $\hat{K}_J(n_1, \dots, n_D) = 1$ for all $(n_1, \dots, n_D) \in \Delta(J)$ and $\hat{K}_J(n_1, \dots, n_D) = 0$ whenever some n_ℓ lies outside of $\Delta_{j_\ell - 1} \cup \Delta_{j_\ell} \cup \Delta_{j_\ell+1}$. Then $P_J u = K_J * P_J u$, so by Young's inequality we have constants c_m, independent of u, N and Δ_J such that

$$\begin{aligned}\int_{\mathbf{T}^D} |P_J u(\theta)|^{2p-4} \frac{d^D\theta}{(2\pi)^D} &\leq c_1 \big\|K_J\big\|^{2p-4}_{L^{(2p-4)/(p-1)}} \|P_J u\|^{2p-4}_{L^2} \\ &\leq c_2 |\Delta_J|^{p-3} N^{p-2}\end{aligned} \tag{2.15}$$

for all $u \in \Omega_N$. Likewise, we have

$$\int_{\mathbf{T}^D} |P_J v(\theta)|^4 \frac{d^D\theta}{(2\pi)^D} \leq c_3 \|K_J\|_{L^{4/3}}^4 \|P_J u\|^4$$
$$\leq c_4 |\Delta_J| \|P_J v\|_{L^2}^4. \tag{2.16}$$

Hence by the Cauchy–Schwarz inequality, we have

$$\Big(\frac{d^2}{dt^2}\Big)_{t=0} H_{\Delta(J)}(u+tv)$$
$$\geq \Big(\frac{D}{4}|\Delta(J)|^{2/D} - \lambda(p-1)c_5|\Delta(J)|^{(p-2)/2} N^{(p-2)/2}\Big) \int_{\mathbf{T}^D} |P_J v(\theta)|^2 \frac{d^D\theta}{(2\pi)^D}, \tag{2.17}$$

where $2/D \geq (p-2)/2$; so given $N > 0$, we can choose $\lambda > 0$ sufficiently small so that the coefficient in parentheses from (2.17) exceeds $D/8$, for all J. $\square$

3. Application to the cubic periodic Schrödinger equation in 1D

Proposition 2.2 involves an exponent $p = 2 + (4/D)$ which equals the optimal exponent for the focussing NLS by [9, page 6]. Such inequalities on dyadic blocks do not of themselves lead directly to $LSI(\alpha)$ on Ω_N. So in sections 3, 4 and 5, we extend Proposition 2.2 to infinite dimensions. The Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbf{T}} \Big|\frac{\partial u}{\partial \theta}\Big|^2 \frac{d\theta}{2\pi} - \frac{\lambda}{4}\int_{\mathbf{T}} \big|u(\theta)\big|^4 \frac{d\theta}{2\pi} \tag{3.1}$$

may be expressed in terms of the canonical variables (f, g) where $f, g \in L^2([0, 2\pi]; \mathbf{R})$, and the field is $u = f + ig$. Then the canonical equation of motion is the cubic Schrödinger equation

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial \theta^2} - \lambda |u|^2 u, \tag{3.2}$$

periodic in θ. Lebowitz, Rose and Speer [15] considered the Gibbs measures for such partial differential equations, exploiting the formal invariance of $H(u)$ and the number operator $N(u) = \int_{\mathbf{T}} |u(\theta)|^2 d\theta/(2\pi)$ with respect to time under the flow generated by the NLS. Bourgain [6, 9] introduced a Gibbs measure ν for spatially periodic solutions, and established the existence of a flow for almost all initial data in the support of ν.

Let $(\gamma_j, \gamma_j')_{j=-\infty}^{\infty}$ be mutually independent standard Gaussian random variables, so that $\sum_{j=-\infty; j\neq 0}^{\infty} e^{ij\theta}(\gamma_j + i\gamma_j')/j$ defines Brownian loop. Let $\lambda, N > 0$ and introduce the ball Ω_N as in (1.2). Often it will be more convenient to use the real Fourier coefficients a_j, b_j of u as canonical coordinates, where $a_j + ib_j = \int u(\theta) e^{-ij\theta} d\theta/(2\pi)$. There exists $Z(N, \lambda) > 0$ such that

$$\nu(du) = Z(N,\lambda)^{-1} \mathbf{I}_{\Omega_N}(u) \exp\Big(\frac{\lambda}{4}\int_{\mathbf{T}} \big|u(\theta)\big|^4 \frac{d\theta}{2\pi}\Big) \prod_{\theta \in [0,2\pi]} du(\theta), \tag{3.3}$$

defines a probability measure, where as in [15, 6] we define

$$\prod_{\theta\in[0,2\pi]} du(\theta) = \prod_{j=-\infty;j\neq 0}^{\infty} \exp\Big(-\frac{j^2}{2}(a_j^2+b_j^2)\Big)\frac{j^2 da_j db_j}{2\pi}, \tag{3.4}$$

namely the measure induced on L^2 by Brownian loop. The indicator $\mathbf{I}_{\Omega_N}(u)$ restricts the field to the bounded subset Ω_N of L^2, and ensures convergence.

We approximate Ω_N by finite-dimensional phase spaces. Let $P_n : L^2 \to \mathrm{span}\{e^{ij\theta} : j = -n, \dots, n\}$ be the usual Dirichlet projection. Then the Hamiltonian

$$H_n(u) = \frac{1}{2}\int_{\mathbf{T}} \Big|\frac{\partial P_n u}{\partial\theta}\Big|^2 \frac{d\theta}{2\pi} - \frac{\lambda}{4}\int_{\mathbf{T}} \big|P_n u(\theta)\big|^4 \frac{d\theta}{2\pi} \tag{3.5}$$

generates the differential equation

$$i\frac{\partial P_n u}{\partial t} = -\frac{\partial^2 P_n u}{\partial\theta^2} - \lambda P_n\Big(|P_n u|^2 P_n u\Big), \tag{3.6}$$

which is associated with a finite-dimensional phase space $P_n L^2$, and a corresponding Gibbs measure. In terms of the Fourier coefficients, (3.6) is an autonomous ordinary differential equation. Let $\hat{X} = (\Omega_N, \|\cdot\|_{L^2}, \nu)$ be the metric measure space associated with (3.3), and with $X^n = \Omega_N \cap P_n L^2$, let $\hat{X}^n = (X^n, \|\cdot\|_{L^2}, \nu_n)$ be the metric measure space associated with (3.5).

Proposition 3.1 *For $0 \leq \lambda N < 3/(14\pi^2)$, the Gibbs measure for NLS on Ω_N satisfies the logarithmic Sobolev inequality*

$$\int_{\Omega_N} F(x)^2 \log\Big(F(x)^2 \Big/ \int F^2 d\nu\Big)\nu(dx) \leq \frac{2}{\alpha}\int \|\nabla F\|^2_{H^{-1}} d\nu, \tag{3.7}$$

for $\alpha = 1 - (14\pi^2 N\lambda)/3$.

Proof. For $f = \Re u$ and $g = \Im u$, the Hamiltonian is

$$H(f+ig) = \frac{1}{2}\int_{\mathbf{T}} \Big[\Big(\frac{\partial f}{\partial\theta}\Big)^2 + \Big(\frac{\partial g}{\partial\theta}\Big)^2\Big]\frac{d\theta}{2\pi} - \frac{\lambda}{4}\int_{\mathbf{T}} \big[f^2+g^2\big]^2 \frac{d\theta}{2\pi}, \tag{3.8}$$

and we aim to show that this is uniformly convex on Ω_N with respect to the homogeneous Sobolev norm $(\int |f'|^2 \frac{d\theta}{2\pi})^{1/2}$ of $\dot{\mathrm{H}}^1$. We consider $U(f+ig) = \int_{\mathbf{T}} (f^2+g^2)^2 \frac{d\theta}{2\pi}$, which contributes a concave term to the Hamiltonian H. We observe that for $0 < t < 1$ and $f, g, p, q \in \mathrm{H}^1$,

$$\begin{aligned} t\big[f^2+g^2\big]^2 &+ (1-t)\big[p^2+q^2\big]^2 - \big[(tf+(1-t)p)^2 + (tg+(1-t)q)^2\big]^2 \\ &= t(1-t)(f-p)^2\big((1+t+t^2)f^2 + (2+2t-2t^2)fp + (2-t+(1-t)^2)p^2\big) \\ &\quad + t(1-t)(g-q)^2\big((1+t+t^2)g^2 + (2+2t-2t^2)gq + (2-t+(1-t)^2)q^2\big) \\ &\quad + 2t(1-t)(f-p)(g-q)(f+p)((1+t)g+(1-t)q) \\ &\quad + 2t(1-t)(g-q)^2p^2 + 2t(1-t)(f-p)^2(tg+(1-t)q)^2. \end{aligned} \tag{3.9}$$

We have the basic estimates $\int(f^2+g^2)\leq N$, and likewise $\int(p^2+q^2)\leq N$, while the Cauchy–Schwarz inequality gives the bounds

$$\|f-p\|^2_{L^\infty}\leq\frac{\pi^2}{3}\int\Big(\frac{\partial f}{\partial\theta}-\frac{\partial p}{\partial\theta}\Big)^2\frac{d\theta}{2\pi} \tag{3.10}$$

and likewise for $\|g-q\|_{L^\infty}$. We integrate (3.9) over $\mathbf{T}$, and use the L^∞ on each of the differences $f-p$ and $g-q$ and the squared L^2 norm to bound each of the sums; hence we have the bound

$$\begin{aligned}0&\leq tU(f+ig)+(1-t)U(p+iq)-U(tf+(1-t)p+i(tg+(1-t)q))\\ &\leq 28Nt(1-t)\int\Big[\Big(\frac{\partial f}{\partial\theta}-\frac{\partial p}{\partial\theta}\Big)^2+\Big(\frac{\partial g}{\partial\theta}-\frac{\partial q}{\partial\theta}\Big)^2\Big]\frac{d\theta}{2\pi}.\end{aligned} \tag{3.11}$$

We deduce that H is uniformly convex with respect to the norm on $\dot{\mathrm{H}}^1$, with

$$\begin{aligned}&tH(f+ig)+(1-t)H(p+iq)-H(tf+(1-t)p+itg+i(1-t)q)\\ &\qquad\geq t(1-t)\Big(\frac{1}{2}-\frac{28\lambda N\pi^2}{12}\Big)\int\Big[\Big(\frac{\partial f}{\partial\theta}-\frac{\partial p}{\partial\theta}\Big)^2+\Big(\frac{\partial g}{\partial\theta}-\frac{\partial q}{\partial\theta}\Big)^2\Big]\frac{d\theta}{2\pi}.\end{aligned} \tag{3.12}$$

The standard inner product on $L^2(\mathbf{T}^D; d^D\theta/(2\pi)^D;\mathbf{R})$ is unitarily equivalent to the standard inner product on $\ell^2(\mathbf{Z}^D)$ under the Fourier transform, and under this pairing, the dual space of $\mathrm{H}^s(\mathbf{T}^D;\mathbf{C})$ is $\mathrm{H}^{-s}(\mathbf{T}^D;\mathbf{C})$. In particular, the dual space of $\dot{\mathrm{H}}^1(\mathbf{T};\mathbf{R})$ is $\dot{\mathrm{H}}^{-1}(\mathbf{T};\mathbf{R})$. So by Bobkov and Ledoux's Proposition 3.1 of [2], the inequality (3.7) holds for all continuously differentiable $F:X^n\to\mathbf{R}$, which depend on only finitely many Fourier coefficients. Then by Lemma 2.1, we can deduce (3.7) for all F. □

Theorem 3.2 *Let $p=4$, $D=1$ and $0<N\lambda<3/(14\pi^2)$. Then $\hat{X}^\infty$ of the focussing cubic NLS has finite diameter and satisfies $LSI(1-(14\pi^2N\lambda/3))$, and $\hat{X}^n\to\hat{X}^\infty$ in D_{L^2} as $n\to\infty$.*

Proof. This follows from Lemma 2.1 and Proposition 3.1. Note that $\|\nabla F\|_{\mathrm{H}^{-1}}\leq\|\nabla F\|_{L^2}$, so (3.7) implies (1.3). □

Remark. One can extend the L^2 convergence result in Theorem 3.2 to all $\lambda,N>0$, although the proof becomes more complicated.

4. Periodic Zakharov system in 1D

Let $u(\theta,t)$ and $n(\theta,t)$ be periodic in the space θ variable; here u is the complex electrostatic envelope field and n is the real ion density fluctuation field. Then the periodic Zakharov model is the pair of coupled differential equations

$$\begin{aligned}i\frac{\partial u}{\partial t}&=-\frac{\partial^2 u}{\partial\theta^2}+nu;\\ \frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial\theta^2}&=\frac{\partial^2}{\partial\theta^2}\big(|u|^2\big).\end{aligned} \tag{4.1}$$

The initial condition is

$$u(\theta,0)=\varphi(\theta),\quad n(\theta,0)=a(\theta),\quad \frac{\partial n}{\partial t}(\theta,0)=b(\theta); \tag{4.2}$$

and Bourgain [7] established global existence of solutions of (4.1) for initial data $\varphi \in \mathrm{H}^1$, $a \in L^2$ and $b \in \mathrm{H}^{-1}$. We now introduce V as the solution of

$$\begin{aligned} \frac{\partial V}{\partial \theta} &= \frac{\partial n}{\partial t}, \\ \frac{\partial V}{\partial t} &= -\frac{\partial n}{\partial \theta} - \frac{\partial}{\partial \theta}\big(|u|^2\big), \end{aligned} \tag{4.3}$$

such that $\int_{\mathbf{T}} V(\theta,t)\,\frac{d\theta}{2\pi} = 0$; existence may be verified from Fourier series. Then we introduce the Hamiltonian

$$H(u,n) = \frac{1}{4}\int_{\mathbf{T}} \Big(2\Big|\frac{\partial u}{\partial \theta}\Big|^2 - |u|^4 + \big(n+|u|^2\big)^2 + V^2\Big)\,\frac{d\theta}{2\pi}, \tag{4.4}$$

which suggests that we introduce further variables $\tilde{n} = (n+|u|^2)/\sqrt{2}$ and $W = (d/d\theta)^{-1}V/\sqrt{2}$. The canonical variables which lead to the system (4.3) are $(\Re u, \Im u)$ and $(n, \sqrt{2}W)$. Then H and $\int_{\mathbf{T}} |u|^2\,\frac{d\theta}{2\pi}$ are invariant under the flow, so we can restrict attention to Ω_B as in (1.2) with $D=1$. Then the Gibbs measure on $\Omega_B \times L^2(\mathbf{T};\mathbf{R}) \times L^2(\mathbf{T};\mathbf{R})$ is defined by

$$\begin{aligned} \nu(du d\tilde{n} dW) = Z^{-1}\Big[\mathbf{I}_{\Omega_B}(u)\exp\Big(\frac{1}{4}\int_{\mathbf{T}}|u|^4\frac{d\theta}{2\pi} - \frac{1}{2}\int_{\mathbf{T}}\Big|\frac{\partial u}{\partial\theta}\Big|^2\frac{d\theta}{2\pi}\Big)\prod_{\theta\in\mathbf{T}} d^2u(\theta)\Big] \\ \times\Big[\exp\Big(-\frac{1}{2}\int_{\mathbf{T}}\tilde{n}^2\Big)\prod_{\theta\in\mathbf{T}} d\tilde{n}(\theta)\Big]\Big[\exp\Big(-\frac{1}{2}\int_{\mathbf{T}}\Big(\frac{\partial W}{\partial\theta}\Big)^2\frac{d\theta}{2\pi}\Big)\prod_{\theta\in\mathbf{T}} dW(\theta)\Big]. \end{aligned} \tag{4.5}$$

We say that $f : L^2 \to \mathbf{R}$ is a cylindrical function, if there exists a compactly supported smooth function $F : \mathbf{R}^n \to \mathbf{R}$ and $\xi_1, \dots, \xi_n \in L^2$ such that $f(\phi) = F(\langle\phi,\xi_1\rangle, \dots \langle\phi,\xi_n\rangle)$.

Proposition 4.1 *There exists $B > 0$ such that the Gibbs measure for the periodic Zakharov system satisfies a logarithmic Sobolev inequality for all cylindrical functions.*

Proof. The Gibbs measure is the direct product of three measures which satisfy logarithmic Sobolev inequalities, as follows. Let $(\gamma_k)_{k=-\infty}^{\infty}$ be mutually independent standard Gaussian random variables, where γ_k has distribution μ_k on $\mathbf{R}$. Then a typical field $\tilde{n}$ has the form

$$\tilde{n}(\theta,0) = \sum_{k=1}^{\infty}\big(\gamma_k\cos k\theta + \gamma_{-k}\sin k\theta\big), \tag{4.6}$$

which converges in $\mathrm{H}^{-(1/2)-\varepsilon}$ for all $\varepsilon > 0$ almost surely. By results of Gross and Federbush, each μ_k satisfies $LSI(1)$ on $\mathbf{R}$, and likewise $\otimes_{k=-n}^{n}\mu_k$ on Euclicean space. The canonical Gaussian measure on L^2 has the characteristic property that for any finite-dimensional subspace X^n, the orthogonal projection $P_n : L^2 \to X^n$ induces the standard Gaussian probability measure on X^n with respect to the induced Euclidean structure; see [18, page 327]. In particular, this applies to $\otimes_{k=-\infty}^{\infty}\mu_k$ and the subspace $X^n = \mathrm{span}\{\xi_j : j = 1, \dots, n\}$ on which the cylindircal function lives. By [21, page 574; 3] this shows that the middle factor in (4.5) satisfies $LSI(1)$, and there is no need to truncate the domain of the $\tilde{n}$ variable.

Likewise, a typical W field initially has the form $W(\theta,0)=\sum_{k=1}^{\infty}(\gamma_k\cos k\theta+\gamma_{-k}\sin k\theta)/k$ and hence the final factor in (4.5) arises from the direct product of Gaussian measures that satisfy $LSI(1)$ on $\mathbf{R}$; hence we have $LSI(1)$ for this product.

Finally, the first factor in (4.5) is the Gibbs measure ν for NLS with $p=4$, so by Proposition 3.1, ν satisfies $LSI(1/2)$ for $B<3/28\pi^2$. Combining these results, as in [21, page 574; 4], we obtain a logarithmic Sobolev inequality where the gradient is

$$\|\nabla F\|^2=\|\nabla_u F\|^2_{\mathrm{H}^{-1}}+\|\nabla_{\tilde n}F\|^2_{L^2}+\|\nabla_W F\|^2_{\mathrm{H}^{-1}}. \tag{4.7}$$

$\square$

5. Periodic KdV equation in 1D

Consider $u:\mathbf{T}\times(0,\infty)\to\mathbf{R}$ such that $u(\,\cdot\,,t)\in L^2(\mathbf{T})$ for each $t>0$, and introduce the Hamiltonian

$$H(u)=\frac{1}{2}\int_{\mathbf{T}}\Big(\frac{\partial u}{\partial\theta}(\theta,t)\Big)^2\frac{d\theta}{2\pi}-\frac{\lambda}{6}\int_{\mathbf{T}}u(\theta,t)^3\frac{d\theta}{2\pi},$$

where $\lambda>0$ is the reciprocal temperature. Then the canonical equation of motion $\frac{\partial u}{\partial t}=\frac{\partial}{\partial\theta}\frac{\delta H}{\delta u}$ gives the KdV equation

$$\frac{\partial u}{\partial t}=-\frac{\partial^3 u}{\partial\theta^3}-\lambda u\frac{\partial u}{\partial\theta}. \tag{5.1}$$

For a suitably differentiable solution u of (5.1), both $\int u(\theta,t)^2d\theta/2\pi$ and $H(u)$ are invariant with respect to time. On the ball

$$B_N=\Big\{\phi\in L^2(\mathbf{T};\mathbf{R}):\int_{\mathbf{T}}\phi(\theta)^2\frac{d\theta}{2\pi}\le N\Big\} \tag{5.2}$$

with indicator $\mathbf{I}_{B_N}$ one can define a Gibbs measure

$$\nu(d\phi)=Z_N(\lambda)^{-1}\mathbf{I}_{B_N}(\phi)e^{-H(\phi)}\prod_{\theta\in[0,2\pi)}d\phi(\theta) \tag{5.3}$$

where $Z_N(\phi)$ is a normalizing constant, chosen to make $\nu(d\phi)$ a probability measure.

The metric probability space $(\Omega_N,\|\cdot\|_{L^2},\nu)$ arises as the limit of finite-dimensional metric probability spaces, which are defined in terms of random Fourier series. Let $X^n=\{(a_j,b_j)_{j=1}^n\in\mathbf{R}^{2n}:\phi(\theta)=\sum_{j=1}^n a_j\cos j\theta+b_j\sin j\theta\in B_N\}$ where we introduce the trigonometric polynomial $\phi(\theta)=\sum_{j=1}^n(a_j\cos j\theta+b_j\sin j\theta)$ and then the probability measure

$$\nu_n(dadb)=Z_n^{-1}\mathbf{I}_{B_N}(\phi)\exp\Big(\frac{\lambda}{6}\int_{\mathbf{T}}\phi(\theta)^3\frac{d\theta}{2\pi}\Big)\exp\Big(-\sum_{j=1}^n j^2(a_j^2+b_j^2)/2\Big)\prod_{j=1}^n da_jdb_j \tag{5.4}$$

for a suitable $Z_n=Z_n(N,\lambda)>0$. We then let $\hat X^{(n)}=(X^n,\|\cdot\|_{\ell^2},\nu_n)$, which is finite dimensional.

Lemma 5.1 *Suppose that $0 \leq \lambda\sqrt{N} < 3/\pi^2$. Then the Gibbs measure satisfies the logarithmic Sobolev inequality*

$$\int_{\Omega_N} f(x)^2 \log\Big(f(x)^2 / \int f^2 d\nu\Big)\nu(dx) \leq \frac{2}{\alpha}\int_{\Omega_N} \|\nabla f\|^2_{\mathrm{H}^{-1}}\nu(dx) \tag{5.5}$$

where $\alpha = 1 - 3^{-1}\pi^2\lambda\sqrt{N}$.

Proof. A related result was given in [3] with a larger norm on the right-hand side. Here we give a proof that is based upon an observation of Schmuckensläger concerning uniformly convex Hamiltonians [2, Proposition 3.1]. For $0 < t < 1$, we have

$$\begin{aligned} &tH(u) + (1-t)H(v) - H(tu + (1-t)v) \\ &\quad = \frac{t(1-t)}{2}\int_{\mathbf{T}}\Big(\frac{\partial u}{\partial\theta} - \frac{\partial v}{\partial\theta}\Big)^2\frac{d\theta}{2\pi} - \frac{\lambda t(1-t)}{6}\int_{\mathbf{T}}(u-v)^2\big((1+t)u + (2-t)v\big)\frac{d\theta}{2\pi} \end{aligned} \tag{5.6}$$

where the final term is estimated by the Cauchy–Schwarz inequality by

$$\begin{aligned} \Big|\int_{\mathbf{T}}(u-v)^2\big((1+t)u + (2-t)v\big)\frac{d\theta}{2\pi}\Big| &\leq \Big(\int_{\mathbf{T}}(u-v)^4\frac{d\theta}{2\pi}\Big)^{1/2}\Big(\int_{\mathbf{T}}\big((1+t)u + (2-t)v\big)^2\frac{d\theta}{2\pi}\Big)^{1/2} \\ &\leq \pi^2\sqrt{N}\int_{\mathbf{T}}\Big(\frac{\partial u}{\partial\theta} - \frac{\partial v}{\partial\theta}\Big)^2\frac{d\theta}{2\pi}. \end{aligned} \tag{5.7}$$

Hence for $\alpha = 1 - 3^{-1}\lambda\pi^2\sqrt{N} > 0$, we have a uniformly convex H such that

$$tH(u) + (1-t)H(v) - H(tu + (1-t)v) \geq \frac{t(1-t)\alpha}{2}\int_{\mathbf{T}}\Big(\frac{\partial u}{\partial\theta} - \frac{\partial v}{\partial\theta}\Big)^2\frac{d\theta}{2\pi}; \tag{5.8}$$

so H is uniformly convex with respect to $\mathrm{H}^1(\mathbf{T};\mathbf{R})$. □

Theorem 5.2 *Let $0 \leq \lambda\sqrt{N} < 3/\pi^2$. Then $(\Omega_N, \|\cdot\|_{L^2}, \nu)$ of KdV has finite diameter, satisfies $LSI(1 - \pi^2\lambda\sqrt{N}/3)$, and is the limit in D_{L^2} of $\hat{X}^n$ as $n \to \infty$.*

Proof. Theorem 5.2 follows from lemmas 2.1 and 5.1. □

6. Logarithmic Sobolev inequality for critical power $p = 6$ in 1D

Now we consider the critical exponent $p = 6$, and the Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbf{T}}\Big(\frac{\partial u}{\partial\theta}\Big)^2\frac{d\theta}{2\pi} - \frac{\lambda}{6}\int_{\mathbf{T}} u(\theta)^6\frac{d\theta}{2\pi}. \tag{6.1}$$

Lebowitz, Rose and Speer show that for $0 < \lambda \leq 1$, there exists $N_0 > 0$ such that the Gibbs measure for H can be normalized on Ω_N for $N < N_0$, but not for $N > N_0$. To obtain a logarithmic Sobolev inequality, we specialize further and for $1/4 < s < 1/2$ and $\kappa > 0$ let

$$\Omega_{N,\kappa} = \Big\{u \in \mathrm{H}^s(\mathbf{T}) : \int_{\mathbf{T}}|u(\theta)|^2\frac{d\theta}{2\pi} \leq N;\ \sum_{n=-\infty}^{\infty}|n|^{2s}|\hat{u}(n)|^2 \leq \kappa\Big\}. \tag{6.2}$$

Proposition 6.1 *Let $N < N_0$ and $0 < \lambda \leq 1$, and $1/4 < s < 1/2$, then let ν_N be the Gibbs measure on Ω_N associated with potential H.*

(i) The sequence of convex and compact subsets $(\Omega_{N,\kappa})_{\kappa=1}^{\infty}$ of Ω_N is increasing and there exist $\varepsilon, C(\varepsilon) > 0$ such that $\nu_N(\Omega_{N,\kappa}) \geq 1 - C(\varepsilon)e^{-\varepsilon\kappa^2}$.

(ii) Let $\hat{\nu}_N$ be ν_N renormalized on $\Omega_{N,\kappa}$ as a probability. Then for all $\kappa > 0$ there exists $\alpha = \alpha(\kappa, N) > 0$ such that $(\Omega_{N,\kappa}, \|\cdot\|_{L^2}, \hat{\nu}_N)$ satisfies $LSI(\alpha)$.

Proof. (i) Compactness and convexity follow from simple facts about the Fourier multiplier sequence $(|n|^{-2s})$ on L^2. Let μ be the Gaussian measure on L^2 that is induced by Brownian loop. Then by the Cauchy–Schwarz inequality, we have

$$\int_{\Omega_N} \exp(\varepsilon\|u\|^2_{\mathrm{H}^s})\nu_N(du) \leq \frac{\Big(\int_{\Omega_N} \exp(2\varepsilon\|u\|^2_{\mathrm{H}^s})\mu(du) \int_{\Omega_N} \exp(3^{-1}\lambda \int_{\mathbf{T}} u^6)\mu(du)\Big)^{1/2}}{\int_{\Omega_N} \mu(du) \int_{\Omega_N} \exp(6^{-1}\lambda \int_{\mathbf{T}} u^6)\mu(du)}, \quad (6.3)$$

where for suitably small $\varepsilon > 0$ the right-hand side integrals are all finite and together define $C(\varepsilon)$. Then we conclude by applying Chebyshev's inequality.

(ii) For integers $k = 1, 2, \ldots$, let $\Delta_k = \{2^{k-1}, 2^{k-1}+1, \ldots, 2^k - 1\}$ be the k^{th} dyadic interval of integers; for $k < 0$, let $\Delta_k = \{n : -n \in \Delta_{-k}\}$; also let $\Delta_0 = \{0\}$. Next let K_k be de la Vallée Poussin's kernel associated with Δ_k so $\hat{K}_k(n) = 1$ for all $n \in \Delta_k$, and $\hat{K}_k(n) = 0$ for n outside $\Delta_{k-1} \cup \Delta_k \cup \Delta_{k+1}$. Also, let $(\varepsilon_k)_{k=1}^{\infty}$ be the usual Rademacher functions. By the Littlewood–Paley theorem, there exist constants $C_1, C_2 > 0$ *etc.* independent of u such that

$$\|u\|^4_{L^4} \leq C_1 \mathbf{E}\,\Big\|\sum_{k=-\infty}^{\infty} \varepsilon_k K_k * u\Big\|^4_{L^4} \leq C_2 \Big(\sum_{k=-\infty}^{\infty} \big\|K_k * u\big\|^2_{L^4}\Big)^2, \quad (6.4)$$

and we can use Young's inequality to show

$$\big\|K_k * u\big\|_{L^4} \leq C_3 \|K_k\|_{L^{4/3}} \|K_k * u\|_{L^2} \leq C_4 |\Delta_k|^{(1/4)-s} \|u\|_{\mathrm{H}^s}. \quad (6.5)$$

Hence H^s embeds continuously in L^4.

We choose $M > 2N_0^2(40^{4s+1}(2\pi\kappa)^4 3^{-2})^{1/(4s-1)}$ and introduce

$$U(u) = \frac{M}{2} \int_{\mathbf{T}} |u(\theta)|^2 \frac{d\theta}{2\pi}, \quad (6.6)$$

so that U is bounded on Ω with $0 \leq U(u) \leq MN \leq MN_0$. Then we consider the modified Hamiltonian $H(u) + U(u)$, and check that it is uniformly convex, with

$$\begin{aligned}
&\Big(\frac{d^2}{dt^2}\Big)_{t=0} \big(H(u+tv) + U(u+tv)\big) \\
&\qquad = \int_{\mathbf{T}} \Big(\frac{\partial v}{\partial \theta}\Big)^2 \frac{d\theta}{2\pi} - 5\lambda \int_{\mathbf{T}} |u(\theta)|^4 |v(\theta)|^2 \frac{d\theta}{2\pi} + M \int_{\mathbf{T}} |v(\theta)|^2 \frac{d\theta}{2\pi} \\
&\qquad \geq \int_{\mathbf{T}} \Big(\frac{\partial v}{\partial \theta}\Big)^2 \frac{d\theta}{2\pi} - 40\lambda \|v\|^2_{L^\infty} \int_{\mathbf{T}} \Big|\sum_{k=-\infty}^{-n} K_k * u + \sum_{k=n}^{\infty} K_k * u\Big|^4 \frac{d\theta}{2\pi} \\
&\qquad\quad + M \int_{\mathbf{T}} |v(\theta)|^2 \frac{d\theta}{2\pi} - 40\lambda \Big\|\sum_{k=-n+1}^{n-1} K_k * u\Big\|^4_{L^\infty} \int_{\mathbf{T}} |v(\theta)|^2 \frac{d\theta}{2\pi}. \quad (6.7)
\end{aligned}$$

By using the Littlewood–Paley decomposition as above, we obtain the lower bound on (6.7)

$$\big(1-80\lambda\kappa^2|\Delta_n|^{1-4s}\pi^2/3\big)\int_{\mathbf{T}}\Big(\frac{\partial v}{\partial\theta}\Big)^2\,\frac{d\theta}{2\pi}+\big(M-40\lambda|\Delta_n|^2N^2-80\lambda\kappa^2|\Delta_n|^{1-4s}\big)\int_{\mathbf{T}}|v(\theta)|^2\,\frac{d\theta}{2\pi}. \tag{6.8}$$

Now we choose n to be the smallest integer such that $2^n=|\Delta_n|>(160\pi^2\kappa^2/3)^{1/(4s-1)}$, so that the first coefficient in (6.8) exceeds $1/2$, while M was chosen above so that

$$\Big(\frac{d^2}{dt^2}\Big)_{t=0}\big(H(u+tv)+U(u+tv)\big)\geq\frac{1}{2}\int_{\mathbf{T}}\Big(\frac{\partial v}{\partial\theta}\Big)^2\frac{d\theta}{2\pi}+\frac{1}{2}\int_{\mathbf{T}}v(\theta)^2\,\frac{d\theta}{2\pi}, \tag{6.9}$$

and we have uniform convexity. Hence there exists $Z(N)>0$ such that the measure

$$Z(N)^{-1}e^{-H(u)-U(u)}\mathbf{I}_{\Omega_{N,\kappa}}(u)\prod_{\theta\in[0,2\pi]}du(\theta) \tag{6.10}$$

can be normalized and satisfies a logarithmic Sobolev inequality with constant $\alpha_0>0$. The original Gibbs measure appears when we perturb the potential by adding the bounded function U, to remove $-U$; hence by the Holley–Stroock perturbation theorem [11; 21, page 574] ν_N also satisfies a logarithmic Sobolev inequality with constant

$$\alpha\geq\alpha_0\exp(-NM)\geq\alpha_0\exp\big(-2(40^{4s+1}(2\pi\kappa)^4 3^{-2})^{1/(4s-1)}NN_0^2\big). \tag{6.11}$$

□

7. The finite-dimensional Gross–Piatevskii equation in 2D

Let $u\in L^2(\mathbf{T}^2;\mathbf{C})$, and $a_k+ib_k=\hat u(k)$ be the decomposition of the Fourier coefficients into real and imaginary parts. With the canonical variables $(a_k,b_k)_{k\in\mathbf{Z}^2}$, the Hamiltonian

$$H=\frac{1}{2}\int_{\mathbf{T}^2}\|\nabla u\|^2\,\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}-\frac{\lambda}{4}\int_{\mathbf{T}^2}\big(V*|u|^2\big)|u|^2\,\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} \tag{7.1}$$

gives rise to the G-P equation (1.5). The $L^2(\mathbf{T}^2;\mathbf{C})$ norm is invariant under the flow for smooth periodic solutions.

Following Bourgain [8], we introduce a Gibbs measure via random Fourier series as in (1.4) with $D=2$. Now b does not belong to $L^2(\mathbf{T}^2;\mathbf{C})$ almost surely, whereas b defines a distribution in $\mathrm{H}^{-s}(\mathbf{T}^2;\mathbf{C})$ almost surely for all $s>0$. We cannot therefore construct the canonical ensemble in precisely the same way as in sections 3,4 and 5; instead, we need to introduce finite-dimensional approximations for which the L^2 norms depend upon the dimension.

We define the number operator by

$$N_n=\sum_{k=(k_1,k_2)\in\mathbf{Z}^2;|k_1|,|k_2|\leq n}\frac{2}{|k|^2+\rho}, \tag{7.2}$$

so that $N_n\approx 2\log n$ as $n\to\infty$. Then for $N>0$ let Ω_N be as in (1.2) with $D=2$. Let $P_n:L^2(\mathbf{T}^2;\mathbf{C})\to\mathrm{span}\{e^{ik\cdot\theta};k\in\mathbf{Z}^2;k=(k_1,k_2);|k_1|,|k_2|\leq n\}$ be the usual Dirichlet

projection onto the span of the characters indexed by lattice points in the square of side $2n$ centred at the origin. For $B>0$, we let $X^n = P_nL^2 \cap \Omega_{N_n+B}$ with the metric given by the L^2 norm, so that the diameter of X^n increases with dimension. Accordingly, we replace $|u|^2$ in (7.1) by $|u_n|^2 - \kappa(N_n+B)$ where $u_n = P_nu$. This is an instance of Wick renormalization.

In the following computations, we have integrals over $\mathbf{T}^2$ with respect to $d\theta_1 d\theta_2/(2\pi)^2$, and we suppress the variables of integration. Hence we take the Hamiltonian to be

$$H_n(u) = \frac{1}{2}\int_{\mathbf{T}^2} \|\nabla u\|^2 - \frac{\lambda}{4}\int_{\mathbf{T}^2} \big(V * |u|^2\big)|u|^2 + \frac{\lambda}{2}\kappa \hat{V}(0)(N_n+B)\int_{\mathbf{T}^2} |u|^2. \tag{7.3}$$

We can regard X^n as a compact and convex subset of $\mathbf{C}^m$ for some $m \le 4(n+1)^2$, and define the Gibbs measure via

$$\nu_n(dadb) = Z_n^{-1}\mathbf{I}_{\Omega_{N_n+B}}(u)e^{-H_n(u)} \prod_{k=(k_1,k_2)\in \mathbf{Z}^2; |k_1|,|k_2|\le n} da_k db_k, \tag{7.4}$$

for $u = \sum_{k=(k_1,k_2)\in \mathbf{Z}^2; |k_1|,|k_2|\le n} (a_k + ib_k)e^{ik\cdot\theta}$.

Brydges and Slade [10] consider focussing periodic NLS in 2D and show that some standard routes to renormalization are blocked. However, allow the possibility that there exist invariant measures in the case in which $N_n \to \infty$ and $\lambda_n \to 0+$ as $n\to\infty$; see page 489. This is the situation we consider in Proposition 7.1.

Proposition 7.1 *(i) Suppose that $V \in L^2(\mathbf{T}^2;\mathbf{R})$. Then for all $B>0$, there exists $\lambda_n>0$ such that the Gibbs measure ν_n on X^n corresponding to H_n satisfies $LSI(1/2)$, so*

$$\int_{X^n} f(x)^2 \log\Big(f(x)^2\Big/\int f^2 d\nu_n\Big)\nu_n(dx) \le 4\int_{X^n} \|\nabla f\|^2_{\mathrm{H}^{-1}(\mathbf{T}^2)}\nu_n(dx). \tag{7.5}$$

(ii) Suppose further that $V \in L^\infty(\mathbf{T}^2;\mathbf{R})$ and that $\kappa\hat{V}(0) > 3\|V\|_{L^\infty}$. Then for all $B,\lambda>0$ and all n, $(X^n, \|\cdot\|_{L^2}, \nu_n)$ satisfies $LSI(1/2)$.

Proof. We prove that the Hamiltonian is uniformly convex, by introducing

$$\begin{aligned}\Big(\frac{d^2}{dt^2}\Big)_{t=0} H(u+tw) = &\int_{\mathbf{T}^2} \|\nabla w\|^2 + \lambda\kappa\hat{V}(0)(N_n+B)\int_{\mathbf{T}^2}|w|^2\\ &-\frac{\lambda}{2}\int_{\mathbf{T}^2}\big(|w|^2 * V\big)|u|^2 - \frac{\lambda}{2}\int_{\mathbf{T}^2}\big(|u|^2 * V\big)|w|^2\\ &-\frac{\lambda}{2}\int_{\mathbf{T}^2}\big((u\bar{w}+\bar{u}w) * V\big)(u\bar{w}+\bar{u}w).\end{aligned} \tag{7.6}$$

(i) By Young's inequality, we have

$$\int_{\mathbf{T}^2}\big(|w|^2 * V\big)|u|^2 \le \|u\|^2_{L^2}\|V\|_{L^2}\|w\|^2_{L^4}, \tag{7.7}$$

and likewise

$$\int_{\mathbf{T}^2} \big(|w|^2 * V\big)|u|^2 \leq \|u\|_{L^2}^2 \|V\|_{L^2} \|w\|_{L^4}^2; \tag{7.8}$$

while each term in the final term in (7.6) is bounded by Young's inequality and Hölder's inequality, so that

$$\begin{aligned}\int_{\mathbf{T}^2} \big(|uw| * |V|\big)|uw| &\leq \|uw\|_{L^{4/3}} \big\||V| * |uw|\big\|_{L^4} \\ &\leq \|u\|_{L^2}^2 \|V\|_{L^2} \|w\|_{L^4}^2. \end{aligned} \tag{7.9}$$

By the Sobolev embedding theorem, we have $\|w - \int w\|_{L^4} \leq C_4 \|\nabla w\|_{L^2}$, for some $C_4 > 0$. Hence

$$\begin{aligned}\Big(\frac{d^2}{dt^2}\Big)_{t=0} H(u+tw) \geq \Big(1 - 3\lambda C_4 (N_n + B)\|V\|_{L^2}\Big) \int_{\mathbf{T}^2} \|\nabla w\|^2 \\ + \lambda (N_n + B)\big(\kappa \hat{V}(0) - 3C_4 \|V\|_{L^2}\big) \int_{\mathbf{T}^2} |w|^2. \end{aligned} \tag{7.12}$$

By choosing $\lambda > 0$ such that $1/2 > 3\lambda C_4 (N_n + B)\|V\|_{L^2}$, we obtain uniform convexity with constant $\alpha = 1/2$. Then $LSI(1/2)$ follows from [2, Proposition 3.1].

(ii) When V is bounded, we can use Young's inequality to bound

$$\int \big(|u|^2 * V\big)|w|^2 \leq \|V\|_{L^\infty} \|u\|_{L^2}^2 \|w\|_{L^2}^2, \tag{7.11}$$

and likewise for the similar terms in (7.6). Hence we obtain the inequality

$$\begin{aligned}&\Big(\frac{d^2}{dt^2}\Big)_{t=0} H(u+tw) \\ &\quad \geq \int_{\mathbf{T}^2} \|\nabla w\|^2 + \lambda\Big(\kappa \hat{V}(0)(N_n + B) - 3\|V\|_{L^\infty} \int_{\mathbf{T}^2} |u|^2\Big) \int_{\mathbf{T}^2} |w|^2. \end{aligned} \tag{7.12}$$

Again $LSI(\alpha)$ follows from [2, Proposition 3.1]. □

8. The Gross–Piatevskii equation on Sobolev space with negative index

To conclude the paper, we obtain a logarithmic Sobolev inequality for the G-P equation (1.5) on a suitable subset of $\mathrm{H}^{-s}(\mathbf{T}^2; \mathbf{C})$. The convolution

$$|u|^2 * V(\theta) = \sum_{m \in \mathbf{Z}^2} (\widehat{|u|^2})(m) \hat{V}(m) e^{im\cdot\theta} \tag{8.1}$$

in the potential is to be interpreted probabilistically, since $u(\theta) = \sum_{k \in \mathbf{Z}^2 \setminus \{0\}} (\gamma_k + i\gamma_k') e^{ik\cdot\theta}/|k|$ does not define an $L^2(\mathbf{T})$ function almost surely.

For $0 < s < 1/4$, $0 < \varepsilon < 1/8$, $K_1 > 0$ and $K_2 > 5$, let

$$\tilde{\Omega} = \Big\{(a_j)_{j \in \mathbf{Z}^2} \in \mathbf{C}^\infty : \sum_{j \in \mathbf{Z}^2 \setminus \{0\}} |a_j|^2 / |j|^{2+2s} \leq K_1; \quad |a_j| \leq K_2 |j|^{(1/4)-\varepsilon}, \quad \forall j \in \mathbf{Z}^2\Big\}, \tag{8.2}$$

so that $\tilde{\Omega}$ is a convex set. Let $(\gamma_j)_{j\in \mathbf{Z}^2}$ be mutually independent standard complex Gaussian random variables, so that γ_j has distribution μ_j, and let $\tilde{\mu}$ be the product measure $\otimes_{j\in\mathbf{Z}^2}\mu_j$ on $\mathbf{C}^\infty$. Let $J:\ell^2(\mathbf{Z}^2;\mathbf{C})\rightarrow \mathrm{H}^1(\mathbf{T}^2;\mathbf{C})$ be the linear map $J(a_j)=\sum_{j\in\mathbf{Z}^2\setminus\{0\}} a_je^{ij\cdot\theta}/\vert j\vert$, and let

$$\Omega=\Bigl\{u\in \mathrm{H}^{-s}:\Vert u\Vert_{\mathrm{H}^{-s}}\leq K_1;\quad \vert \hat u(j)\vert \leq K_2\vert j\vert^{-(3/4)-\varepsilon},\quad \forall j\in \mathbf{Z}^2\Bigr\}. \tag{8.3}$$

Then J induces a measure μ on H^{-s}, which is mainly supported on Ω.

Theorem 8.1 *Suppose that $V\in \mathrm{H}^{1+2s}(\mathbf{T}^2;\mathbf{R})$ for some $s>0$.*

(i) Then $\mu(\Omega)\rightarrow 1$ as $K_1,K_2\rightarrow\infty$;

(ii) for all K_1,K_2 sufficiently large and $0<\varepsilon<1/8$ there exist $\lambda>0$ and $\alpha>0$ such that the Gibbs measure ν, normalized to be a probability on Ω, satisfies $LSI(\alpha)$, so

$$\int_\Omega f(u)^2\log\Bigl(f(u)^2/\int f^2d\nu\Bigr)\nu (du)\leq {\frac{2}{\alpha}}\int_\Omega \Vert\nabla f\Vert^2_{\mathrm{H}^{-s}}\,\nu (du) \tag{8.4}$$

for all $f\in L^2(\Omega ;\nu ;\mathbf{R})$ that are differentiable with $\Vert \nabla f\Vert_{\mathrm{H}^{-s}}\in L^2(\Omega ;\nu ;\mathbf{R})$.

(iii) The transportation cost for cost function $c(f,g)=\Vert f-g\Vert^2_{\mathrm{H}^{-s}}$ and all $\omega\in \mathrm{Prob}_2(\Omega)$ that are of finite relative entropy with respect to ν satisfies

$$\mathrm{W}_2(\omega ,\nu)^2\leq {\frac{2}{\alpha}}\mathrm{Ent}(\omega\mid \nu). \tag{8.5}$$

Remark. The hypotheses imply that $V\in L^\infty$. In summary, the Gibbs measure produces a metric probability space $(\Omega ,\Vert\cdot\Vert_{\mathrm{H}^{-s}},\nu)$ of finite diameter that satisfies LSI.

Proof. (i) We introduce the event

$$\Gamma =\Bigl\{ \vert\gamma_j\vert\leq K_2\vert j\vert^{(1/4)-\varepsilon},\quad \forall j\in\mathbf{Z}^2\setminus\{0\}\Bigr\}, \tag{8.6}$$

which by mutual independence of the γ_j has measure

$$\begin{aligned}\tilde\mu (\Gamma) &=\prod_{j\in\mathbf{Z}^2\setminus\{0\}}\Bigl(1-2\int_{K_2\vert j\vert^{1/4-\varepsilon}}^\infty e^{-s^2/2}{\frac{ds}{\sqrt{2\pi}}}\Bigr)\\ &\geq \exp\Bigl(-4\sum_{j\in\mathbf{Z}^2\setminus\{0\}}\int_{K_2\vert j\vert^{1/4-\varepsilon}}^\infty e^{-s^2/2}{\frac{ds}{\sqrt{2\pi}}}\Bigr)\end{aligned} \tag{8.7}$$

since $K_2e^{K_2^2/2}>4$. Also by Chebyshev's inequality, we have

$$\begin{aligned}\tilde\mu\Bigl\{\sum_{j\in\mathbf{Z}^2\setminus\{0\}}{\frac{\vert\gamma\vert^2}{\vert j\vert^{2s+2}}}\geq K_1^2\Bigr\} &\leq e^{-K_1^2/4}\prod_{j\in\mathbf{Z}^2\setminus\{0\}}\Bigl(1-{\frac{1}{2\vert j\vert^{2+2s}}}\Bigr)^{-1/2}\\ &\leq \exp\Bigl(-{\frac{K_1^2}{4}}+\sum_{j\in\mathbf{Z}^2\setminus\{0\}}{\frac{1}{2\vert j\vert^{2s+2}}}\Bigr);\end{aligned} \tag{8.8}$$

so by estimating these sums by the Euler–Maclaurin sum formula, we obtain

$$\tilde{\mu}(\tilde{\Omega}) \geq \exp\Big(-\frac{2(6+\pi)e^{-K_2^2/2}}{K_2\sqrt{2\pi}}\Big) - \exp\Big(\frac{-K_1^2}{4} + \frac{\pi}{2s} + 5\Big), \tag{8.9}$$

hence $\tilde{\mu}(\tilde{\Omega}) \to 1$ as $K_1, K_2 \to \infty$.

(ii) By results of Gross and Federbush, each μ_j satisfies $LSI(1)$ for the standard gradient and distance over $\mathbf{C}$; hence their direct product $\tilde{\mu}$ satisfies $LSI(1)$ on $\tilde{\Omega}$, where the norm of the gradient is computed in the norm of ℓ^2. Lemma 2.1 enables us to pass from finite to infinite dimensions. We prove below that there exist $\kappa > 0$ and $Z > 0$ such that $\tilde{\nu}(da) = Z^{-1}e^{U(J(a))}\tilde{\mu}(da)$ defines a probability measure on $\tilde{\Omega}$ such that

$$\int_{\tilde{\Omega}} \exp\Big(\kappa\big\|\nabla(U\circ J)(a)\big\|_{\ell^2}^2\Big)\tilde{\nu}(da) < \infty. \tag{8.10}$$

Then $\tilde{\nu}$ satisfies $LSI(\alpha)$ for some $\alpha > 0$ by the condition of Aida and Shigekawa [1]; see also [21, Remark 21.5]. Letting $u = J(a)$ and $v = J(b)$, we have

$$\big\langle\nabla(U\circ J)(a), b\big\rangle_{\ell^2} = (d/dt)_{t=0}U\circ J(a+tb) = \int_{\mathbf{T}^2} \frac{\delta U}{\delta u}(\theta)v(\theta)\,\frac{d^2\theta}{(2\pi)^2}, \tag{8.11}$$

while the norms satisfy

$$\begin{aligned}\big\|\nabla(U\circ J)(a)\big\|_{\ell^2} &= \sup\Big\{\big|\langle\nabla(U\circ J)(a), b\rangle\big| : \|b\|_{\ell^2} \leq 1\Big\}\\ &= \sup\Big\{\Re\int_{\mathbf{T}^2} \frac{\delta U}{\delta u}(\theta)v(\theta)\,\frac{d^2\theta}{(2\pi)^2} : v = J(b); \|b\|_{\ell^2} \leq 1\Big\}\\ &\leq \Big\|\frac{\delta U}{\delta u}\Big\|_{\mathrm{H}^{-s}},\end{aligned} \tag{8.12}$$

since $J : \ell^2 \to \mathrm{H}^s$ defines a contractive linear operator for $0 < s < 1$, and H^s is the dual of H^{-s} under the integral pairing.

Let ν be the measure on Ω that is induced from $\tilde{\nu}$ on $\tilde{\Omega}$ by J, then normalized to be a probability. Then we obtain the logarithmic Sobolev inequality for the Gibbs measure

$$\begin{aligned}\int_\Omega f(\phi)^2\log\Big(f(\phi)^2/\int f^2d\nu\Big)\nu(d\phi) &= \int_{\tilde{\Omega}} f(J(a))^2\log\Big(f(J(a))^2/\int f\circ Jd\tilde{\nu}\Big)e^{U(J(a))}\tilde{\mu}(da)/Z\\ &\leq \frac{2}{\alpha}\int_{\tilde{\Omega}}\big\|\nabla(f\circ J)(a)\big\|_{\ell^2}^2\tilde{\nu}(da)\\ &\leq \frac{2}{\alpha}\int_\Omega\big\|\nabla f(\phi)\big\|_{\mathrm{H}^{-s}}^2\nu(d\phi),\end{aligned} \tag{8.13}$$

where the final step follows as in (8.12).

So this leaves us with the task of verifying (8.10). The Hamiltonian involves

$$U(u) = \frac{\lambda}{4}\int_{\mathbf{T}^2}\Big(\big(|u|^2 - \int|u|^2\big)*V\Big)|u|^2\frac{d^2\theta}{(2\pi)^2} \tag{8.14}$$

with gradient

$$\langle \nabla U(u), v\rangle = \Big(\frac{d}{dt}\Big)_{t=0} U(u+tv) \tag{8.15}$$
$$= \frac{\lambda}{4}\int_{\mathbf{T}^2}\Big[\Big(\big(|u|^2 - \int |u|^2\big) * V\Big)(u\bar v + v\bar u) + \Big((u\bar v + \bar u v) * V\Big)|u|^2\Big]\frac{d^2\theta}{(2\pi)^2}.$$

The integrand involves the Fourier series

$$\big(|u|^2 * V\big)u = \sum_{m\in\mathbf{Z}^2} \widehat{\big(|u|^2\big)}(m)\hat V(m)\sum_{j\in\mathbf{Z}^2}\hat u(j)e^{i(j+m)\cdot\theta}, \tag{8.16}$$

where $(1+|j+m|)(1+|m|)\geq (1+|j|)$, so for all $u\in\Omega$ we have

$$\Big\|\sum_{j\in\mathbf{Z}^2\setminus\{0\}}\hat u(j)e^{i(j+m)\cdot\theta}\Big\|_{\mathrm{H}^{-s}} \leq \Big(\sum_{j\in\mathbf{Z}^2\setminus\{0\}}\frac{|\hat u(j)|^2}{|j|^{2s}}\Big)^{1/2}|m|^s \leq K_1|m|^s, \tag{8.17}$$

hence

$$\Big\|\big(|u|^2 * V\big)u\Big\|_{\mathrm{H}^{-s}} \leq K_1\sum_{m\in\mathbf{Z}^2}|m|^s|\hat V(m)|\big|\widehat{\big(|u|^2\big)}(m)\big|. \tag{8.18}$$

To estimate the right-hand side of (8.18), we will later use the following lemma.

Lemma 8.2 *(i) The $\widehat{(|u|^2)}(m)$ are uniformly exponentially square integrable over Ω with respect to μ, so there exist $C_1,\kappa>0$ such that*

$$\int_\Omega \exp\Big(\kappa^2\big|\widehat{(|u|^2)}(m)\big|^2\Big)\mu(du) < C_1 \qquad (m\in\mathbf{Z}^2\setminus\{0\}). \tag{8.19}$$

(ii) A similar statement holds for ν on Ω, possibly with different constants.

Proof. (i) We have $\widehat{(|u|^2)}(-m) = \sum_j(\gamma_j + i\gamma_j')(\gamma_{j+m} - i\gamma'_{j+m})/|j||j+m|$, so we require to bound $\sum_{r=1}^\infty d_r^{(m)}$ where each $d_r^{(m)}$ is a sum over an annulus

$$d_r^{(m)} = \sum_{j\in\mathbf{Z}^2\setminus\{0,-m\};r-1<|j|\leq r}\frac{\gamma_j\gamma_{j+m}}{|j||j+m|}. \tag{8.20}$$

Observe that on $\tilde\Omega$ the random variables γ_j are symmetric and we can independently replace each γ_j by $\pm\gamma_j$, without affecting the distribution of $\tilde\mu$ on $\tilde\Omega$.

The sequence $(d_r^{(m)})$ is multiplicative in the sense of [12] so that for all strictly increasing subsequences $r_1<r_2<\ldots<r_n$ of integers,

$$\int_{\tilde\Omega} d_{r_1}^{(m)}d_{r_2}^{(m)}\ldots d_{r_n}^{(m)}\,\tilde\mu(d\gamma) = 0. \tag{8.21}$$

To see this, consider a product of terms, with one taken from the sum (8.20) for each factor $d_{r_j}^{(m)}$ and consider the lattice points ℓ that index the γ_ℓ from factors in this product. In

particular, consider ℓ such that the distance from the origin is a maximum, and observe that this is attained at some point of the form $j+m$, and that γ_{j+m} appears only once in the product, hence integrates to give zero.

Observe also that $|d_r^{(m)}| \leq \delta_r$ where $\delta_r = C_0 K_2^2 r^{-(1/2)-\varepsilon}$ for some universal constant C_0, so that $\delta_r \leq 3C_0^2 K_2^4/8\varepsilon$ as follows: The most challenging case is when $|m| = r$, and we can compare $\delta_r \leq K_2^2 r^{-(3/4)-\varepsilon} \sum_{j\in \mathbf{Z}^2\setminus\{0,-m\}; r-1\leq |j|<r} |j+m|^{-(3/4)-\varepsilon}$ with the sum arising with the lattice points j replaced by points equally spaced around the circle of centre the origin and radius r, which produces the integral $K_2^2 r^{-(1/2)-2\varepsilon} \int_0^{2\pi} |\sin(\theta/2)|^{-(3/4)-\varepsilon} d\theta$.

Bounded multiplicative systems satisfy similar concentration inequalities to bounded martingale differences as in [20]. By Jakubowski and Kwapien's [12] contraction principle, for any convex function $\Phi : \mathbf{R}^n \to [0,\infty)$, the inequality

$$\tilde\mu(\tilde\Omega)^{-1} \int_{\tilde\Omega} \Phi(d_1^{(m)}, \dots, d_n^{(m)})\, d\tilde\mu \leq \mathbf{E}\Phi(\delta_1\varepsilon_1, \dots, \delta_n\varepsilon_n) \tag{8.22}$$

holds, where $(\varepsilon_j)_{j=1}^\infty$ is the usual sequence of mutually independent Rademacher functions. In particular, choosing $\kappa > 0$ so that $\kappa^2 3C_0^2 K_2^4/8\varepsilon < 1$, we have

$$\begin{aligned}
\tilde\mu(\tilde\Omega)^{-1} \int_{\tilde\Omega} \exp\Big(\frac{\kappa^2}{2}\big(\sum_{r=1}^n d_r^{(m)}\big)^2\Big) d\tilde\mu &\leq \int_{-\infty}^\infty \mathbf{E}\exp\Big(t\sum_{r=1}^n \kappa\delta_r\varepsilon_r\Big) \exp(-t^2/2)\frac{dt}{\sqrt{2\pi}} \\
&= \int_{-\infty}^\infty \prod_{r=1}^n \cosh(\kappa\delta_r t)\, \exp(-t^2/2)\frac{dt}{\sqrt{2\pi}} \\
&\leq \int_{-\infty}^\infty \exp\Big(\frac12\sum_{r=1}^n \kappa^2\delta_r^2 t^2 - \frac{t^2}{2}\Big)\frac{dt}{\sqrt{2\pi}} \\
&= \Big(1-\kappa^2\sum_{r=1}^n \delta_r^2\Big)^{-1/2}.
\end{aligned} \tag{8.23}$$

Letting $n \to \infty$ and applying Fatou's lemma, we obtain (8.19).

(ii) This follows from (i) by Hölder's inequality. □

Conclusion of the proof of Theorem 8.1. (ii) We need to deduce (8.10) from (8.19). We introduce $C_3 > 0$ such that $1/C_3 \leq K_1(\pi/s+10)$ such that $C_3 \sum_{m\in\mathbf{Z}^2\setminus\{0\}} |m|^{-2-2s} = 1$, and then use Hölder's inequality to obtain

$$\begin{aligned}
\int_\Omega e^{\kappa U(u)}\mu(du) \leq \prod_{m\in\mathbf{Z}^2\setminus\{0\}} \Big[\Big(\int_\Omega \exp\Big[\kappa\big|\widehat{(|u|^2)}(m)\big|^2/C_3\Big]\mu(du)\Big)^{C_3|\hat V(m)|/2} \\
\times \Big(\int_\Omega \exp\Big[\kappa\big|\widehat{(|u|^2)}(-m)\big|^2/C_3\Big]\mu(du)\Big)^{C_3|\hat V(m)|/2}\Big]
\end{aligned} \tag{8.24}$$

By Lemma 7.3, all of these integrals converge for sufficiently small $\kappa > 0$, so the Gibbs measure $d\nu = e^U d\mu$ can be normalized on Ω to define a probability measure which is absolutely continuous with respect to μ.

We can introduce $C(s) \geq (\pi/s+10)^{-1}$ such that $\sum_{j\in\mathbf{Z}^2\setminus\{0\}} C(s)/|j|^{2+2s}=1$, and then we separate $\hat V$ from $\widehat{|u|^2}$ by Cauchy–Schwarz inequality, before applying Hölder's inequality to obtain

$$\begin{aligned}&\int_\Omega \exp\Big[\kappa_0^2\Big(\sum_{m\in\mathbf{Z}^2\setminus\{0\}} |m|^s|\hat V(m)|\big|(\widehat{|u|^2})(m)\big|\Big)^2\Big]\mu(du)\\ &\leq \prod_{m\in\mathbf{Z}^2\setminus\{0\}}\Big(\int_\Omega \exp\Big[\frac{\kappa_0^2}{C(s)}\sum_{j\in\mathbf{Z}^2\setminus\{0\}}|j|^{2+4s}|\hat V(j)|^2\big|\,(\widehat{|u|^2})(m)\big|^2\Big]\mu(du)\Big)^{C(s)/|m|^{2+2s}}. \quad (8.25)\end{aligned}$$

By taking $\kappa_0>0$ sufficiently small, we can ensure that all the integrals and the product converge. This confirms that (8.10) holds, and hence gives the logarithmic Sobolev inequality.

(iii) The transportation inequality follows from the logarithmic Sobolev inequality (8.4) as in [21, Theorem 22.17]. □

Let $X^n=\mathrm{span}\{e^{ij\cdot\theta}: j\in\mathbf{Z}^2; |j|\leq n\}$ be the subspace of $L^2(\mathbf{T}^2;\mathbf{C})$ that is spanned by the characters that are indexed by the lattice points in the disc with radius n, and let $P_n: L^2(\mathbf{T}^2;\mathbf{C})\to X^n$ be the orthogonal projection. Let ν_n be the Gibbs measure

$$\nu_n(du)=Z_n^{-1}\mathbf{I}_\Omega(u)\exp\big(U(P_nu)\big)\prod_{m\in\mathbf{Z}^2\setminus\{0\}} e^{-|m|^2(a_m^2+b_m^2)/2}da_m db_m/2\pi. \quad (8.26)$$

Let ω_n be the marginal distribution of ν_n on X^n.

Corollary 8.3 *The* $(X^n\cap\Omega, \|\cdot\|_{\mathrm{H}^{-s}},\omega_n)$ *converge in* D_{L^2} *to* $(\Omega,\|\cdot\|_{\mathrm{H}^{-s}},\nu)$ *as* $n\to\infty$.

Proof. (i) First we prove that $U(P_nu)\to U(u)$ almost surely and in L^2 with respect to μ on Ω as $n\to\infty$. The difference in the potentials has a Fourier expansion

$$\begin{aligned}U(P_nu)-U(u)&=\int (V*|P_nu|^2)|P_nu|^2-\int(V*|u|^2)|u|^2\\ &=\sum_m \hat V(m)\Big((\widehat{|P_nu|^2})(m)-(\widehat{|u|^2})(m)\Big)\Big((\widehat{|u|^2})(-m)\Big)\\ &\quad+\sum_m \hat V(m)\Big((\widehat{|P_nu|^2})(m)\Big)\Big((\widehat{|P_nu|^2})(-m)-(\widehat{|u|^2})(-m)\Big); \quad (8.27)\end{aligned}$$

hence

$$\begin{aligned}&\big|U(P_pu)-U(P_nu)\big| \quad (8.28)\\ &\leq 2\sum_{m\in\mathbf{Z}^2}|\hat V(m)|\big|(\widehat{|P_pu|^2})(m)-(\widehat{|P_nu|^2})(m)\big|\Big(\big|(\widehat{|P_pu|^2})(m)-(\widehat{|P_nu|^2})(m)\big|+\big|(\widehat{|u|^2})(m)\big|\Big)\end{aligned}$$

where

$$\big|(\widehat{|P_pu|^2})(m)-(\widehat{|P_nu|^2})(m)\big|\leq\Big|\sum_{r=n+1}^{\ell} d_r^{(m)}\Big|. \quad (8.29)$$

We observe that $(\widehat{|P_nu|^2})(m)$ has a similar expansion to (8.31), except that only those j with $|j|\leq n$ contribute; so Lemma 8.2; hence $(\widehat{|P_nu|^2})(m)$ satisfies similar estimates to $(\widehat{|u|^2})(m)$, with the same constants.

Let $\Phi : \mathbf{C}^{\ell-n}\rightarrow [0,\infty)$ be the convex function

$$\Phi(z_1,\dots,z_{\ell-n})=\max_p\Bigl\{\Bigl|\sum_{t=n}^{p}z_{t-n}\Bigr|^4 : n\leq p\leq \ell\Bigr\} \tag{8.30}$$

associated with the fourth power of maximal partial sums. Then by the contraction principle from [12], the martingale maximal theorem in L^4 and Khinchine's inequality we have

$$\begin{aligned}\Bigl(\int_\Omega \Phi(d_n^{(m)},\dots,d_\ell^{(m)})\mu(du)\Bigr)^{1/4} &\leq \Bigl(\mathbf{E}_\varepsilon \Phi(\delta_n\varepsilon_n,\dots,\delta_\ell\varepsilon_\ell)\Bigr)^{1/4}\\ &\leq \frac{4\sqrt{2}}{3}\Bigl(\sum_{p=n}^{\ell}\delta_p^2\Bigr)^{1/2}\\ &\leq \frac{4\sqrt{2}C_0K_2^2}{3\sqrt{\varepsilon n^{\varepsilon}}}. \end{aligned}\tag{8.31}$$

The sequence $(\hat{V}(m))_{m\in\mathbf{Z}^2}$ is summable, so we deduce from (8.28) via the triangle inequality in $L^2(\mu)$ and Hölder's inequality that

$$\begin{aligned}&\Bigl(\int_\Omega \max_p\Bigl\{|U(P_pu)-U(P_nu)|^2 : n\leq p\leq \ell\Bigr\}\mu(du)\Bigr)^{1/2}\\ &\quad\leq 4\sum_{m\in\mathbf{Z}^2}\bigl|\hat{V}(m)\bigr|\Bigl(\int_\Omega \Phi(d_n^{(m)},\dots,d_\ell^{(m)})\mu(du)\Bigr)^{1/4}\\ &\qquad\times\Bigl(\int_\Omega\bigl|(\widehat{|u|^2})(m)\bigr|^4\mu(du)+\int_\Omega \Phi(d_n^{(m)},\dots,d_\ell^{(m)})\mu(du)\Bigr)^{1/4};\end{aligned}\tag{8.32}$$

and hence by (8.31)

$$\mu\Bigl\{\max_p\bigl\{|U(P_pu)-U(P_nu)|^2 : n\leq p\leq \ell\bigr\}\geq\delta\Bigr\}\rightarrow 0 \qquad (\delta>0) \tag{8.33}$$

as $\ell\geq n\rightarrow\infty$, so $U(P_nu)\rightarrow U(u)$ almost surely and in $L^2(\mu)$ as $n\rightarrow\infty$.

(ii) We have

$$\mathrm{Ent}(\nu_n\mid\nu)=\int_\Omega\Bigl(U(P_nu)-U(u)+\log Z-\log Z_n\Bigr)\nu_n(du), \tag{8.34}$$

where the normalizing constants satisfy $\liminf_{n\rightarrow\infty}Z_n\geq Z$, and the preceding arguments show that $\int_\Omega|U(P_nu)-U(u)|^2\mu(du)\rightarrow 0$ as $n\rightarrow\infty$ and $\int_\Omega e^{2U(P_nu)}\mu(du)\leq C$. Hence $\mathrm{Ent}(\nu_n\mid\nu)\rightarrow 0$ as $n\rightarrow\infty$. By the transportation inequality (8.5), this implies $\mathrm{W}_2(\nu_n,\nu)\rightarrow 0$

as $n \to \infty$. Essentially, ν_n is the tensor product of ω_n with a Gaussian measure on H^{-s} with variance that converges to zero as $n \to \infty$; indeed, the tail of the product (8.26) satisfies

$$\int \sum_{m \in \mathbf{Z}^2; |m| \geq n} \frac{a_m^2 + b_m^2}{|m|^{2s}} \prod_{m \in \mathbf{Z}^2; |m| \geq n} e^{-|m|^2(a_m^2+b_m^2)/2} \frac{|m|^2 da_m db_m}{2\pi} = \sum_{m \in \mathbf{Z}^2; |m| \geq n} \frac{2}{|m|^{2+2s}}$$
$$\leq \frac{4\pi}{s(n-1)^{2s}}.$$

Hence $\mathrm{D}_{L^2}((X^n, \|\cdot\|_{\mathrm{H}^{-s}}, \omega_n), (\Omega_n, \|\cdot\|_{\mathrm{H}^{-s}}, \nu_n)) \to 0$ as $n \to \infty$ as in [19, Example 3.8]. $\square$

Let $\Delta = \partial^2/\partial\theta_1^2 + \partial^2/\partial\theta_2^2$, and write

$$\Phi(u)(\theta, t) = \int_0^t e^{i(t-\tau)\Delta}\Big(\big(|u|^2 * V\big)u\Big)(\theta, \tau)\, d\tau. \tag{8.35}$$

In Proposition 8.4, we verify that the solution of the G-P equation

$$\begin{aligned} -i\frac{\partial u}{\partial t} &= \Delta u + \big(|u|^2 * V\big)u, \\ u(\theta, 0) &= \phi(\theta) \end{aligned} \tag{8.36}$$

with $\phi \in \Omega \subset \mathrm{H}^s(\mathbf{T}^2; \mathbf{C})$ is given by $u = u_0 + w$, where $u_0(\theta, t) = e^{it\Delta}\phi(\theta)$ is the solution of the free periodic Schrödinger equation with initial datum ϕ in the support of Brownian loop on H^{-s} and $w \in \mathrm{H}^s$ is a fixed point of $w \mapsto \Phi(u_0 + w)$.

We say that $f : \mathrm{H}^{-s} \to \mathbf{R}$ is a cylindrical function, if there exists a compactly supported smooth function $F : \mathbf{R}^n \to \mathbf{R}$ and $\xi_1, \dots, \xi_n \in \mathrm{H}^s$ such that $f(\phi) = F(\langle \phi, \xi_1 \rangle, \dots \langle \phi, \xi_n \rangle)$. The following may be compared with Bourgain's results from [9, p. 132].

Proposition 8.4 *Let $0 < s < 1/68$, and let $V \in \mathrm{H}^{\delta+2s+3/2}(\mathbf{T}^2; \mathbf{R})$ for some $\delta > 0$ have $\hat{V}(0) = 0$. Then for all $\eta > 0$, there exists $\Omega_\eta \subseteq \Omega$ and $L_\eta, t_\eta > 0$ such that $\mu(\Omega_\eta) > 1 - \eta$ and*

(i) for all $\phi \in \Omega_\eta$ and $u_0(\theta, \tau) = e^{i\tau\Delta}\phi(\theta)$, the function $\Phi(u_0) \in C([0, T]; \mathrm{H}^s(\mathbf{T}^2; \mathbf{C}))$ for $T > 0$ almost surely;

(ii) $w \mapsto \Phi(u_0 + w)$ is L_η-Lipschitz on bounded subsets of $C([0, T]; \mathrm{H}^s(\mathbf{T}^2; \mathbf{C}))$;

(iii) the Cauchy problem (8.36) has a solution $u(\theta, t)$ for $t \in [0, t_\eta]$ for all $\phi \in \Omega_\eta$;

(iv) $\phi(\theta) \mapsto u(\theta, t)$ for $\phi \in \Omega_\eta$ induces a measure on H^{-s} which satisfies the T_1 transportation inequality, and is invariant in the sense that all cylindrical functions satisfy

$$\int_{\Omega_\eta} f(u(\,\cdot\,, t))\nu(d\phi) = \int_{\Omega_\eta} f(\phi)\nu(d\phi) \qquad (0 \leq t < t_\eta). \tag{8.37}$$

Proof. (i) We write $\|a\|_* = 1 + \|a\|$. Note that (Ω, μ) is invariant under the operation $\phi(\theta) \mapsto e^{i\tau\Delta}\phi(\theta)$. The integral (8.35) may be expressed in Fourier coefficients as

$$\begin{aligned} &\Phi(u_0)(\theta, t) \\ &= \sum_{m \in \mathbf{Z}^2 \setminus \{0\}} \Big[\sum_{j,k : j+k=m} \hat{\phi}(j)\overline{\hat{\phi}(-k)} \frac{e^{it(\|\ell\|^2 - \|\ell+m\|^2)} - 1}{i(\|\ell\|^2 - \|\ell+m\|^2 + \|j\|^2 - \|k\|^2)} \Big] \hat{V}(m) \sum_{\ell} e^{i(\ell+m)\cdot\theta}\hat{\phi}(\ell), \end{aligned} \tag{8.38}$$

and we split this sum into four cases, according to the values of j and k in the inner sum, and then according to ℓ and m in the outer sums. First we note that in the inner sum in square brackets $\|j\|^2-\|k\|^2=(2j-m)\cdot m$, so we split the index set as $\{(j,k)\in\mathbf{Z}^2\times\mathbf{Z}^2 : j+k=m\}=G(\ell,m)\sqcup B(\ell,m)$ where

$$G(\ell,m)=\Bigl\{(j,k): j+k=m; \bigl|\|\ell\|^2-\|m+\ell\|^2+(2j-m).m\bigr|\geq 2^{-2}\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|\Bigr\}, \tag{8.39}$$

and the complementary set

$$B(\ell,m)=\Bigl\{(j,k): j+k=m; \bigl|\|\ell\|^2-\|m+\ell\|^2+(2j-m).m\bigr|< 2^{-2}\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|\Bigr\}, \tag{8.40}$$

so that $B(\ell,m)$ is the set of integral lattice points in a strip in $\mathbf{R}^2$ which has axis perpendicular to m and width $\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|$. Now the sum

$$\sum_{(j,k)\in G(\ell,m)}\frac{\hat\phi(j)\overline{\hat\phi(-k)}(\|\ell\|^2-\|\ell+m\|^2)}{\|\ell\|^2-\|\ell+m\|^2+\|j\|^2-\|k\|^2} \tag{8.41}$$

is exponentially square integrable by Lemma 8.2. Then we take the complementary contribution to the inner sum of (8.38) to be

$$\begin{aligned}
&\Bigl|\sum_{(j,k)\in B(\ell,m)}\frac{\hat\phi(j)\overline{\hat\phi(-k)}}{1+\bigl|\|\ell\|^2-\|\ell+m\|^2+\|j\|^2-\|k\|^2\bigr|}\Bigr|\\
&\leq\sum_{(j,k)\in B(\ell,m)}\frac{K_2^2}{\|j\|_*^{\varepsilon+3/4}\|k\|_*^{\varepsilon+3/4}\bigl|\|\ell\|^2-\|\ell+m\|^2+\|j\|^2-\|k\|^2\bigr|}\\
&\leq\frac{K_2^2}{\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|^{1/16}}\sum_{(j,k)\in B(\ell,m)}\Bigl\{\frac{1}{\|j\|_*^{\varepsilon+1/2}\|m-j\|^{\varepsilon+1/2}}\Bigr\}\\
&\quad\times\Bigl(\frac{1}{\|j\|_*^{1/8}\|m-j\|_*^{1/8}\bigl|\|\ell\|^2-\|\ell+m\|^2+2j\cdot m-\|m\|^2\bigr|^{15/16}}\Bigr).
\end{aligned}\tag{8.42}$$

Then we split $j=j_\perp+j_m$, where $j_\perp$ is perpendicular to m, and j_m parallel to m; the sum in braces is dominated by the corresponding sum over $j_\perp$ and is bounded; while the sum in parentheses is dominated by the corresponding sum over j_m and is also bounded; so the whole expression (8.42) is

$$\leq C\frac{K_2^2}{\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|^{1/16}}. \tag{8.43}$$

We deduce that for all $\eta>0$, there exist a subset $\Omega_\eta\subset\Omega$ with $\mu(\Omega_\eta)>1-\eta$ and a constant C_η such that

$$\begin{aligned}
\bigl\|\Phi(u_0)\bigr\|_{\mathrm{H}^s}&\leq C_\eta\sum_m|\hat V(m)|\Bigl\|\sum_\ell\frac{e^{i(m+\ell)\cdot\theta}\hat\phi(\ell)}{\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|^{1/16}}\Bigr\|_{\mathrm{H}^s}\\
&\leq C_\eta\sum_m\|\hat V(m)|\|m\|_*^{2s}\Bigl[\sum_\ell\Bigl(\frac{\|\ell+m\|^{2s}\|\ell\|_*^{2s}}{\|m\|_*^{2s}\bigl|\|\ell\|^2-\|\ell+m\|^2\bigr|^{1/8}}\Bigr)\frac{|\hat\phi(\ell)|^2}{\|\ell\|_*^{2s}}\Bigr]^{1/2}.
\end{aligned}\tag{8.44}$$

We split this sum into a sum over the index set

$$A = \Big\{(\ell, m) \in \mathbf{Z}^2 \times \mathbf{Z}^2 : \big|\|\ell\|^2 - \|\ell + m\|^2\big|^{1/8} \geq \|\ell\|_*^{4s}\Big\}$$

and a sum over the complementary set A^c. On A, the factor in parentheses from (8.44) is bounded, so the upper bound $\sum_m |\hat{V}(m)|\|m\|^{2s}\|\phi\|_{\mathrm{H}^{-s}}$ is immediate. On A^c, we use the bound $|\hat{\phi}(j)| \leq K_2\|j\|_*^{-\varepsilon-3/4}$, and for each m, we compare the sum over $(\ell, m) \in A^c$ with an integral in polar coordinates (r, ψ) over the region

$$\big\{(r, \psi) \in (1, \infty) \times (-\pi, \pi) : 2\|m\|r|\sin\psi| \leq \|m\|^2 + r^{32s}\big\}; \tag{8.45}$$

so we have a bound on $\sum_{\ell \in A^c}$ of

$$\begin{aligned}\sum_{\ell : |\|\ell\|^2 - \|\ell+m\|^2| < \|\ell\|^{32s}} & \frac{K_2^2}{\|m\|_*^{2s}|\|\ell\|^2 - \|m+\ell\|^2|^{1/8}\|\ell\|_*^{2\varepsilon+3/2}} \\ & \leq 2K_2^2 \int_1^\infty r^{2s-3/2-2\varepsilon} \int_0^{(r^{32s}+\|m\|^2)/2r\|m\|} d\psi\, r dr \\ & \leq 2CK_2\Big(\|m\| + \frac{1}{\|m\|}\Big). \end{aligned} \tag{8.46}$$

The series $\sum_m |\hat{V}(m)|\|m\|^{2s+1/2}$ converges, so $\Phi(u_0)$ belongs to $C([0,T]; \mathrm{H}^s)$.

(ii) In this proof, we use concentration of measure to prove Lipschitz continuity of a function; this reverses the usual flow of the theory as in [5, 21]. For v and w in the unit ball of $C([0,T]; \mathrm{H}^s(\mathbf{T}^2; \mathbf{C}))$, We have

$$\begin{aligned} &\Phi(v + u_0)(\theta, t) - \Phi(w + u_0)(\theta, t) \\ &\quad = \int_0^t e^{i(t-\tau)\Delta}\Big(\big(|u_0|^2 * V\big)(v - w)\Big)(\theta, \tau)\, d\tau \\ &\quad + \int_0^t e^{i(t-\tau)\Delta}\Big(\big((|v|^2 + \bar{v}u_0 + v\bar{u}_0) * V\big)(v - w)\Big)(\theta, \tau)\, d\tau \\ &\quad + \int_0^t e^{i(t-\tau)\Delta}\Big(\big(((v - w)\bar{v} + w(\bar{v} - \bar{w}) + u_0(\bar{v} - \bar{w}) + \bar{u}_0(v - w)) * V\big)w\Big)(\theta, \tau)\, d\tau. \end{aligned} \tag{8.47}$$

In the final integral, we can use the simple bound

$$\big|u_0\widehat{(\bar{v} - \bar{w})}(m)\big| \leq \big\|u_0\big\|_{\mathrm{H}^{-s}}\big\|v - w\big\|_{\mathrm{H}^s} \leq K_1\big\|v - w\big\|_{\mathrm{H}^s}, \tag{8.48}$$

and similar bounds on the other terms; the terms in the middle integral are treated similarly. The first integral, we use the probabilistic estimate of Lemma 8.2: for all $\eta > 0$ there exist $L_\eta > 0$ and a subset $\Omega_\eta \subseteq \Omega$ such that $\mu(\Omega_\eta) > 1 - \eta$ and

$$\sum_m \big|(\widehat{|u_0|^2})(m)\big||\hat{V}(m)|\|m\|^{2s} \leq L_\eta \qquad (u_0(\theta, 0) \in \Omega_\eta),$$

so there exists $C > 0$ such that

$$\sup_{0<t<T} \|\Phi(u_0+v)(\theta,t) - \Phi(u_0+w)(\theta,t)\|_{\mathrm{H}^s} \leq CT(1+L_\eta) \sup_{0<\tau<T} \|v(\theta,\tau) - w(\theta,\tau)\|_{\mathrm{H}^s}. \quad (8.49)$$

(iii) By (i), we have $T > 0$ such that $K_0 = \sup_{0<t<T} \|\Phi(u_0)(\theta,t)\|_{\mathrm{H}^s}$ is finite for all $\phi \in \Omega_\eta$. Now by (8.49) we can shrink the time interval to $[0, t_\eta]$ where $0 < t_\eta < T$, and ensure that

$$B_\eta = \Big\{ w \in C([0,t_\eta]; \mathrm{H}^s(\mathbf{T}^2; \mathbf{C})); \sup_{0<t<t_\eta} \|w(\theta,t)\|_{\mathrm{H}^s} \leq 2K_0 \Big\} \quad (8.50)$$

contains $\Phi(u_0)$ and $w \mapsto \Phi(u_0 + w)$ is (1/2)-Lipschitz on B_η. Indeed, we have

$$\begin{aligned} \sup_{0<t<t_\eta} \|\Phi(u_0+w)(\theta,t)\|_{\mathrm{H}^s} &\leq \sup_{0<t<t_\eta} \|\Phi(u_0+w)(\theta,t) - \Phi(u_0)(\theta,t)\|_{\mathrm{H}^s} + \sup_{0<t<t_\eta} \|\Phi(u_0)(\theta,t)\|_{\mathrm{H}^s} \\ &\leq 2^{-1} \sup_{0<t<t_\eta} \|w(\theta,t)\|_{\mathrm{H}^s} + K_0 \\ &\leq 2K_0. \end{aligned} \quad (8.51)$$

By Banach's fixed point theorem, there exists $w \in B_\eta$ such that $w = \Phi(u_0+w)$; thus we obtain a solution $u(\theta,t) = u_0(\theta,t) + w(\theta,t)$ of G-P (8.36) for $0 < t < t_\eta$.

(iv) We do not assert that $\phi \mapsto \Phi(u_0 + v)$ is Lipschitz; hence we need an indirect proof of (iv) instead of deducing it from Theorem 8.1. The fixed point w satisfies $\|w(\,\cdot\,,t)\|_{\mathrm{H}^s} \leq 2\|\Phi(u_0)(\,\cdot\,,t)\|_{\mathrm{H}^s}$, hence

$$\|u(\,\cdot\,,t)\|_{\mathrm{H}^{-s}} \leq \|\phi\|_{\mathrm{H}^{-s}} + \|\Phi(u_0)(\,\cdot\,,t)\|_{\mathrm{H}^s} \quad (8.52)$$

so there exists $\kappa > 0$ such that

$$\int_{\Omega_\eta} \exp\big(\kappa \|u(\,\cdot\,,t)\|^2_{\mathrm{H}^{-s}}\big) \nu(d\phi) \quad (8.53)$$

is finite. Hence the measure induced on H^{-s} from μ on Ω_η by $\phi \mapsto u(\,\cdot\,,t)$ satisfies a T_1 transportation inequality by Bobkov and Götze's criterion, as in [21, Theorem 22.10].

Let u_n be the solution of the *GP* equation with finite-dimensional Hamiltonian H_n as in (7.3) and initial data $\phi_n(\theta) = \sum_{k:0<|k|\leq n} e^{ik\cdot\theta}(\gamma_k + i\tilde{\gamma}_k)/\|k\|$, and we regard $u_n(\,\cdot\,,t)$ as a random variable for $\phi \in \Omega_\eta$. We have

$$\begin{aligned} &\|u(\,\cdot\,,t) - u_n(\,\cdot\,,t)\|_{\mathrm{H}^{-s}} \\ &\quad \leq 2\|\phi - \phi_n\|_{\mathrm{H}^{-s}} + 2\Big\| \int_0^t e^{i(t-\tau)\Delta} \Big(\big(|w_n + e^{i\tau\Delta}\phi_n|^2 * V\big) \big(e^{i\tau\Delta}\phi_n - e^{i\tau\Delta}\phi\big) \Big) d\tau \Big\|_{\mathrm{H}^{-s}} \\ &\qquad + 2\Big\| \int_0^t e^{i(t-\tau)\Delta} \Big(\big(|w_n + e^{i\tau\Delta}\phi|^2 - |w_n + e^{i\tau\Delta}\phi_n|^2 \big) * V \big) \big(w + e^{i\tau\Delta}\phi\big) \Big) d\tau \Big\|_{\mathrm{H}^{-s}}. \end{aligned} \quad (8.54)$$

As in (iii), one can show that u_n converges to u in the sense that

$$\int_{\Omega_\eta} \|u_n(\,\cdot\,,t) - u(\,\cdot\,,t)\|^2_{\mathrm{H}^{-s}} \mu(d\phi) \to 0 \tag{8.55}$$

as $n \to \infty$. By Liouville's theorem applied to H_n, the corresponding Gibbs measure on phase space is invariant under the flow generated by the canonical equations of motion. Hence by Corollary 8.3, we have weak convergence of the Gibbs measures, so

$$\begin{aligned}\int_{\Omega_\eta} f(u(\,\cdot\,,t))\nu(d\phi) &= \lim_{n\to\infty} \int_{\Omega_\eta} f(u_n(\,\cdot\,,t))\nu_n(d\phi)\\ &= \lim_{n\to\infty} \int_{\Omega_\eta} f(\phi)\nu_n(d\phi)\\ &= \int_{\Omega_\eta} f(\phi)\nu(d\phi). \end{aligned}\tag{8.56}$$

$\square$

Acknowledgement. The author thanks Graham Jameson and Daniel Elton for providing some estimates.

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